NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4281

SECOND-ORDER SLENDER-BODY THEORY - AXISYMMETRIC FLOW

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SUMMARY

Slender-body theory for subsonic and supersonic flow past bodies of revolution is extended to a second approximation. Methods are developed for handling the difficulties that arise at round ends. Comparison is made with experiment and with other theories for several simple shapes.

INTRODUCTION

Slender-body theory is the useful approximation introduced into fluid mechanics by Munk (ref. 1) for calculating the lift of airships, and extended to slender lifting wings in compressible flow by Jones (ref. 2). For such problems concerned with lift, its simplicity is such that the solution is independent of Mach number, and is found merely by solving Laplace's equation in two dimensions.

The theory becomes only slightly more complicated when the thickness of a body is of concern. Then the solution includes a logarithmic term proportional to cross-sectional area that varies with Mach number, as was shown by Ward (ref. 3) in the case of supersonic flow past general slender shapes. The analogous result for subsonic flow was found independently by Keune (ref. 4), Heaslet and Lomax (ref. 5), and Adams and Sears (ref. 6).

Because slender-body theory is so simple and useful, it is natural to attempt to improve its accuracy by including nonlinear effects in higher approximations. Thus, for bodies of revolution in supersonic flow, Lighthill (ref. 7) found the second-order slender-body solution for the crossflow due to incidence, and Broderick (ref. 8) attacked the flow at zero angle of attack. Recently Lighthill has outlined the second approximation for supersonic flow past general shapes (ref. 9). The only application to noncircular shapes is the solution for the elliptic cone at zero incidence (ref. 10). These four references constitute the literature on this subject, aside from papers by Adams and Sears (ref. 6), Legras (ref. 11), and Keune (ref. 12), who ignore nonlinear effects and seek only a closer approach to the full linearized solution.

The present paper is devoted to second-order slender-body theory in subsonic as well as supersonic flow, and is restricted to bodies of revolution. These are the simplest and most practical shapes, and serve to illustrate the methods that will be required later in treating bodies of general cross section. Only zero angle of attack is considered because Lighthill's treatment of the crossflow at supersonic speeds is entirely satisfactory, and could readily be extended to subsonic speeds. On the other hand, Broderick's solution for the present problem of zero incidence at supersonic speeds is so enormously more complicated than necessary that it could probably never be applied to any shape except the cone.

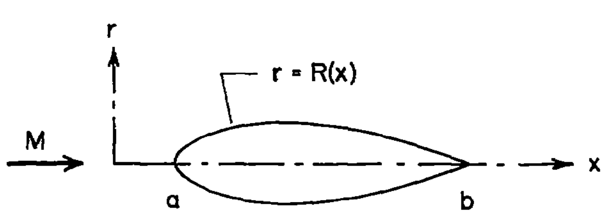
The formal theory set forth here is relatively simple, being comprised in equations (1) to (13). Complications appear, however, in the case of stagnation points, to which a considerable portion of the paper is devoted. It is shown that real difficulties arise only for round noses, and that for subsonic flow they can be overcome by comparison with the known solution for a paraboloid. Only the region spanned by the body is considered, though the flow upstream and downstream could be treated in the same way. The second approximation, like the first, depends upon an integral that is the counterpart for slender bodies of revolution of the "airfoil integral" of subsonic thin-wing theory (ref. 13).

This investigation was begun in 1953, inspired by a suggestion of Max. Heaslet, to whom the author is indebted also for subsequent helpful discussions. Some of the main results were presented at colloquia at the University of Manchester and Fort Halstead in 1954 and 1955. Completion has been delayed by the search for a method of treating round noses, which was only recently found (ref. 14).

FORMAL SECOND APPROXIMATION

Resumé of Second-Order Problem

Consider a uniform subsonic or supersonic stream flowing past a slender body of revolution at zero angle of attack (sketch (a)). The question of just how smooth and slender it must be will be considered later, but the nose (and, in subsonic flow, also the tail), if not pointed, is assumed to be no blunter than round.



Sketch (a).- Notation for body of revolution.

Vorticity affects the flow only in the sixth approximation, and below that the velocity disturbances induced by the body (referred to the speed U of the free stream) are the gradient of a perturbation potential Φ. Linearized theory is concerned with a first approximation φ that satisfies the Prandtl-Glauert equation

$$(1 - M^2)\varphi_{xx} + \varphi_{rr} + \frac{\varphi_r}{r} = 0 \tag{1}$$

(Principal symbols are defined in appendix A.) If one attempts to improve the linearized solution, the second approximation ϕ must satisfy the iteration equation (ref. 15)

$$(1 - M^2)\phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} = M^2\left[2\left(\frac{\gamma + 1}{2} M^2 + 1 - M^2\right)\varphi_x\varphi_{xx} + 2\varphi_r\varphi_{xr} + \varphi_r^2\varphi_{rr}\right] \tag{2}$$

The boundary conditions are that the perturbation potential vanish radially far from the body (actually at the bow wave in supersonic flow), and that the flow be tangent at the surface. To first- and second-order accuracy, this tangency condition is

$$\varphi_r = R' \qquad \text{at} \quad r = R(x) \tag{3a}$$

$$\phi_r = (1 + \varphi_x)R' \qquad \text{at} \quad r = R(x) \tag{3b}$$

With the velocity potential determined, the pressure coefficient is given to second order by

$$C_p = -2\phi_x - \phi_r^2 - (1 - M^2)\varphi_x^2 + M^2\varphi_x\varphi_r^2 + \frac{1}{4} M^2\varphi_r^4 \tag{4}$$

In the slender-body approximation the first term in equation (1) can be neglected, except insofar as it appears in the distant boundary condition. Similarly, for second-order slender-body theory, various terms in equation (2) can be omitted (ref. 9). However, this simplification is unnecessary here because a particular integral of equation (2) itself is known; and it would actually complicate the distant boundary condition.

Resumé of First-Order Slender-Body Solution

Slender-body theory is a further simplification beyond linearization that describes the flow only in the immediate vicinity of the body - more precisely, within a distance from the axis of the order of the local body radius. It can therefore be extracted from the linearized solution by a limiting process. Similarly, the second-order slender-body theory sought here represents the first two terms of an asymptotic series, and can be extracted from the full second-order solution.

For the first-order slender-body solution we adopt the procedure of Keune (ref. 4) and Heaslet and Lomax (ref. 5) as being simpler and more physically appealing than the methods of Fourier and Laplace transformation. The appropriate solution of the linearized equation (1) that vanishes far from the body is

$$\varphi(x,r) = \begin{cases} -\dfrac{1}{2}\displaystyle\int_a^b \dfrac{F(\xi)d\xi}{\sqrt{(x-\xi)^2+\beta^2r^2}}, & \beta \equiv \sqrt{1-M^2} \text{ for subsonic flow} \quad (5a) \\ -\displaystyle\int_a^{x-Br} \dfrac{F(\xi)d\xi}{\sqrt{(x-\xi)^2-B^2r^2}}, & B \equiv \sqrt{M^2-1} \text{ for supersonic flow} \quad (5b) \end{cases}$$

This may be regarded as resulting from a distribution along the axis of the body of sources of strength proportional to a function $F(x)$ that is to be determined from the tangency condition. Differentiating and integrating with respect to x gives[1]

$$\varphi(x,r) = \begin{cases} -\dfrac{1}{2}\dfrac{\partial}{\partial x}\displaystyle\int_a^b F(\xi)\sinh^{-1}\dfrac{x-\xi}{\beta r}\,d\xi & (6a) \\ -\dfrac{\partial}{\partial x}\displaystyle\int_a^{x-Br} F(\xi)\cosh^{-1}\dfrac{x-\xi}{Br}\,d\xi & (6b) \end{cases}$$

Then approximating asymptotically for small r in the integrand (and also in the upper limit for supersonic flow) gives, near the body,

$$\varphi(x,r) = \begin{cases} -\dfrac{1}{2}\dfrac{\partial}{\partial x}\displaystyle\int_a^b F(\xi)\operatorname{sgn}(x-\xi)\ln\dfrac{2|x-\xi|}{\beta r}\,d\xi = F(x)\ln\dfrac{\beta r}{2} - \dfrac{1}{2}\dfrac{\partial}{\partial x}\displaystyle\int_a^b F(\xi)\operatorname{sgn}(x-\xi)\ln|x-\xi|\,d\xi & (7a) \\ -\dfrac{\partial}{\partial x}\displaystyle\int_a^x F(\xi)\ln\dfrac{2(x-\xi)}{Br}\,d\xi = F(x)\ln\dfrac{Br}{2} - \dfrac{\partial}{\partial x}\displaystyle\int_a^x F(\xi)\ln(x-\xi)\,d\xi & (7b) \end{cases}$$

This is the result of Heaslet and Lomax (ref. 5).

Alternative forms of the integrals that are a great deal simpler for either analytical or numerical evaluation were given by Schultz-Piszachich (ref. 16). Excluding an infinitesimal neighborhood of the point $x = \xi$ from the range of integration, carrying out the differentiation indicated in equations (7), adding and subtracting a logarithmic term, and then allowing the excluded neighborhood to vanish leads to

[1]If the body has pointed ends (so that $F = 0$ there), the procedure can be simplified, and it is only necessary to integrate by parts. However, we contemplate treating round ends also, at least in subsonic flow.

$$\varphi(x,r) = \begin{cases} F(x)\ln \dfrac{\beta r}{2\sqrt{(x-a)(b-x)}} + \dfrac{1}{2}\displaystyle\int_a^b \dfrac{F(x) - F(\xi)}{|x - \xi|}\, d\xi & \text{(8a)} \\[2ex] F(x)\ln \dfrac{Br}{2(x-a)} + \displaystyle\int_a^x \dfrac{F(x) - F(\xi)}{x - \xi}\, d\xi & \text{(8b)} \end{cases}$$

The superiority of these forms is clear if $F(x)$ is a polynomial; then the integrands in equations (8) are simply polynomials, whereas those in equations (7) contain logarithms.

Imposing the first-order tangency condition of equation (3a) now determines the source strength $F(x)$ in terms of the body radius $R(x)$ as

$$F(x) = R(x)R'(x)$$

Hence the first-order slender-body solution has the form

$$\varphi = F(x)\ln r + G(x) \tag{9a}$$

where

$$F(x) = R(x)R'(x) \tag{9b}$$

$$G(x) = \begin{cases} F(x)\ln \dfrac{\beta}{2\sqrt{(x-a)(b-x)}} + \dfrac{1}{2}\displaystyle\int_a^b \dfrac{F(x) - F(\xi)}{|x - \xi|}\, d\xi & \text{(9c)} \\[2ex] F(x)\ln \dfrac{B}{2(x-a)} + \displaystyle\int_a^x \dfrac{F(x) - F(\xi)}{x - \xi}\, d\xi & \text{(9d)} \end{cases}$$

The pressure coefficient on the surface of the body is given by

$$C_{p_s} = -2[F'(x)\ln R(x) + G'(x)] - R'^2(x) \tag{10}$$

Second Approximation

The slender-body solution of equation (9a) is clearly a solution of Laplace's equation in the cross plane, which is the Prandtl-Glauert equation (1) with the term $(1 - M^2)\varphi_{xx}$ omitted (except insofar as it is implicit in the boundary condition far from the body). This linear term must therefore be taken into account in the second approximation in addition to the nonlinear terms on the right-hand side of the iteration equation (2). Hence the slender-body iteration equation is

$$\phi_{rr} + \frac{\phi_r}{r} = -(1 - M^2)\varphi_{xx} + M^2\left[2\left(\frac{\gamma+1}{2} M^2 + 1 - M^2\right)\varphi_x\varphi_{xx} + 2\varphi_r\varphi_{xr} + \varphi_r{}^2\varphi_{rr}\right] \tag{11}$$

A particular integral for the linear term on the right, which vanishes far from the body, is given by[2]

$$-\frac{1}{4}(1 - M^2)r^2\varphi_{xx} = \frac{1}{4}(M^2 - 1)r^2(F''\ln r + G'' - F'')$$

A particular integral for the nonlinear terms is known to be given by (ref. 15)

$$M^2\left[\varphi_x\left(\varphi + \frac{\gamma+1}{2}\frac{M^2}{M^2-1} r\varphi_r\right) - \frac{1}{4} r\varphi_r{}^3\right]$$
$$= M^2\left[(F'\ln r + G')\left(F \ln r + G + \frac{\gamma+1}{2}\frac{M^2}{M^2-1} F\right) - \frac{1}{4}\frac{F^3}{r^2}\right]$$

The complete second approximation is the sum of these two particular integrals, plus a complementary solution that will have just the form of the first approximation, equations (9). Hence the second-order slender-body solution for the perturbation potential is

$$\phi(x,r) = (F+f)\ln r + (G+g) + \frac{1}{4}(M^2 - 1)r^2(F''\ln r + G'' - F'') +$$
$$M^2\left[(F'\ln r + G')(F \ln r + G + NF) - \frac{1}{4}\frac{F^3}{r^2}\right] \tag{12a}$$

where

$$N = -n = \frac{\gamma+1}{2}\frac{M^2}{M^2-1} \tag{12b}$$

Here $f(x)$ is the second-order increment in source strength. Imposing the tangency condition of equation (3b) determines it as

$$f(x) = (1 - 2M^2)FF'\ln R - M^2NFF' + (1 - M^2)FG' - M^2F'G -$$
$$\frac{1}{2} M^2 \frac{F^3}{R^2} + \frac{1}{2}(1 - M^2)R^2\left(F''\ln R + G'' - \frac{1}{2} F''\right) \tag{12c}$$

[2]This result can also be obtained by retaining secondary as well as leading terms of the expansions in any of the conventional derivations of slender-body theory. If the Heaslet-Lomax method is followed, it is necessary to differentiate and integrate with respect to x, as in going from equations (5) to (6), two more times in order to avoid divergent integrals.

Then $g(x)$ is related to $f(x)$ in the same way that $G(x)$ is to $F(x)$:

$$g(x) = \begin{cases} f(x) \ln \dfrac{\beta}{2\sqrt{(x-a)(b-x)}} + \dfrac{1}{2} \displaystyle\int_a^b \dfrac{f(x) - f(\xi)}{|x - \xi|} \, d\xi & \text{(12d)} \\[2ex] f(x) \ln \dfrac{B}{2(x-a)} + \displaystyle\int_a^x \dfrac{f(x) - f(\xi)}{x - \xi} \, d\xi & \text{(12e)} \end{cases}$$

However, it will be seen later that this formal result fails at round ends. The proper expression for g for round ends will be given in equation (40) for incompressible flow and equations (52) for subsonic flow.

On the surface of the body the expression of equation (4) for the pressure coefficient can be simplified, using the tangency conditions of equations (3), to

$$C_{p_s} = -2\phi_{x_s} - R'^2\left[1 + (2 - M^2)\varphi_{x_s}\right] + (M^2 - 1)\varphi_{x_s}{}^2 + \frac{1}{4} M^2 R'^4 \tag{13}$$

Slender-Body Integrals

The second-order, like the first-order slender-body solution, is seen to require only the evaluation of the "slender-body integrals"

$$I_a^b\left\{F(x)\right\} \equiv \int_a^b \frac{F(x) - F(\xi)}{|x - \xi|} \, d\xi \qquad \text{(subsonic)} \tag{14a}$$

$$J_a^x\left\{F(x)\right\} \equiv \int_a^x \frac{F(x) - F(\xi)}{x - \xi} \, d\xi \qquad \text{(supersonic)} \tag{14b}$$

and their first three derivatives with respect to x, which involve integrals of the same form:

$$I_a^{b\,'} = \int_a^b \frac{F'(x) - F'(\xi)}{|x - \xi|} \, d\xi + \frac{F(x) - F(a)}{x - a} + \frac{F(b) - F(x)}{b - x} \tag{15a}$$

$$J_a^{x\,'} = \int_a^x \frac{F'(x) - F'(\xi)}{x - \xi} \, d\xi + \frac{F(x) - F(a)}{x - a} \tag{15b}$$

etc.

Note that only a single integral is actually involved, since the subsonic one is related to the supersonic one by

$$I_a^b\left\{F(x)\right\} = J_a^x\left\{F(x)\right\} + J_b^x\left\{F(x)\right\} , \qquad a < x < b \tag{16}$$

However, it is convenient to list both.

As with the analogous airfoil integral of subsonic thin-wing theory, these integrals can be evaluated analytically for a wide variety of functions. A short table is given in appendix B.

Transonic Approximation

It will be seen in later examples that the analytic form of the second approximation is rather complicated even for simple shapes. A further approximation that yields considerable analytic simplification and a remarkably elegant result is that of the transonic small-disturbance theory. In that approximation one retains of the nonlinear terms only the one that is dominant near Mach number unity, so that the full equation of motion is simply

$$(1 - M^2)\phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} = (\gamma + 1)\phi_x\phi_{xx} \tag{17}$$

In plane flow the accuracy is improved by keeping a factor M^2 in the right-hand term, but a test with the exact solution for cones in supersonic flow suggests that the advantage does not persist in axisymmetric flow. The effect of retaining the M^2 is simply to change $(\gamma + 1)$ to $M^2(\gamma + 1)$ in all that follows.

If one attempts to solve this simplified equation by iteration on slender-body theory, the second approximation is, from equations (12),

$$\phi(x,r) = (F + f)\ln r + (G + g) + \frac{\gamma + 1}{2(M^2 - 1)} F(F'\ln r + G') \tag{18a}$$

where

$$F = RR' \tag{18b}$$

$$f = - \frac{\gamma + 1}{2(M^2 - 1)} FF' \tag{18c}$$

and $G(x)$ and $g(x)$ are related to $F(x)$ and $f(x)$ by equations (9c) and (9d) and (12d) and (12e).

This result can be simplified because the two second-order terms in $ln\ r$ cancel. The second-order increment in velocity potential is thus found to be a function only of x, given by

$$\Delta_2\phi \equiv \phi - \varphi = \begin{cases} \dfrac{\gamma+1}{4(1-M^2)}\left[\displaystyle\int_a^b \frac{F(x)-F(\xi)}{|x-\xi|}F'(\xi)d\xi + \frac{F(a)F(x)}{x-a} - \frac{F(b)F(x)}{b-x} - \frac{1}{2}\frac{F^2(a)}{x-a} + \frac{1}{2}\frac{F^2(b)}{b-x}\right], & M<1 \quad (19a) \\ -\dfrac{\gamma+1}{2(M^2-1)}\left[\displaystyle\int_a^x \frac{F(x)-F(\xi)}{x-\xi}F'(\xi)d\xi + \frac{F(a)F(x)}{x-a}\right], & M>1 \quad (19b) \end{cases}$$

Here a correction for round ends that will be derived later (eqs. (52)) has been included in the subsonic case as the last two terms in the bracket of equation (19a). A corresponding correction should probably be found also for the supersonic case; if so, equation (19b) does not apply to a round nose and the last term might as well be omitted.

This incremental potential may be regarded as representing a plane wave standing normal to the body axis whose amplitude is independent of radius within the slender-body approximation (although it of course attenuates at distances large compared with the local body radius, where that approximation fails). It can be shown that this result holds also for bodies of general cross section, where $F(x)$ in equations (19) is $A'(x)/2\pi$ if $A(x)$ is the cross-sectional area.

The surface pressure coefficient is given, in the approximation of transonic small-disturbance theory, by

$$C_{p_s} = -2\phi_x - R'^2 \tag{20}$$

Near round ends in subsonic flow the first-order pressure coefficient is infinite like x^{-1}, and the second-order increment like x^{-2} (so that neither is integrable for drag). The same is true for a round nose in supersonic flow (except that, as just remarked, the second-order increment given above may not be even formally correct).

If round ends are excluded, the drag in supersonic flow is integrable, and the second-order increment is given by

$$\frac{\Delta_2 D}{\frac{1}{2}\rho_\infty U^2} = 2\pi\frac{\gamma+1}{M^2-1}\left[F(b)\int_a^b \frac{F(b)-F(\xi)}{b-\xi}F'(\xi)d\xi - \frac{1}{2}\int_a^b\int_a^b \frac{F(x)-F(\xi)}{x-\xi}F'(x)F'(\xi)d\xi\,dx\right] \tag{21a}$$

Just as in plane flow past an airfoil, this may be either positive or negative, according as the body is fatter near its nose or tail. For if round tails are also excluded, setting $F(x) \to F(-x)$ shows that reversing the direction of flow changes the sign of the drag increment of equation (21a). This means, in particular, that the supersonic drag increment is zero for a body with fore-and-aft symmetry.

The corresponding expression for subsonic flow (with round ends excluded so that the drag is integrable) is

$$\frac{\Delta_2 D}{\frac{1}{2}\rho_\infty U^2} = \pi \frac{\gamma + 1}{1 - M^2} \int_a^b \int_a^b \frac{F(x) - F(\xi)}{|x - \xi|} F'(x)F'(\xi)d\xi \, dx \tag{21b}$$

This differs from the second term in equation (21a) only in having the absolute value signs. As a consequence, however, it can be shown that this drag increment is zero in conformity with D'Alembert's principle.

Oswatitsch and Berndt have shown (ref. 17) that the transonic small-disturbance approximation together with the slender-body approximation implies a similarity rule for surface pressures on affinely related axisymmetric bodies of thickness ratio τ, according to which

$$\frac{C_{p_s}}{\tau^2} + ln(\tau^2|1 - M^2|) = P\left[\frac{M^2 - 1}{(\gamma + 1)\tau^2}\right] \tag{22}$$

Here P is some function of the transonic similarity parameter $(M^2 - 1)/(\gamma + 1)\tau^2$. The present theory gives the first two terms in an asymptotic expansion of the function P for large values of its argument.

EXAMPLES IN SUPERSONIC FLOW

Restrictions on Body Shape

It is implied in the slender-body approximation (as in linearized theory) that the velocity disturbances induced by the body are everywhere small. This imposes serious limitations on the smoothness of the body, even in the first approximation. Thus for bodies of revolution not only must the slope R' be small and continuous, but the curvature R'' as well. Supersonic noses must be at least pointed (R' small), and supersonic tails and subsonic noses and tails must actually be cusped ($R' = 0$).[3]

It is well known in thin-wing theory that the conditions for linearization may be violated locally without destroying the validity

[3]These restrictions are somewhat more severe than those suggested by Ward (ref. 21). He permits discontinuities in curvature and pointed subsonic ends, but it is readily verified that these both lead to logarithmically infinite surface pressures.

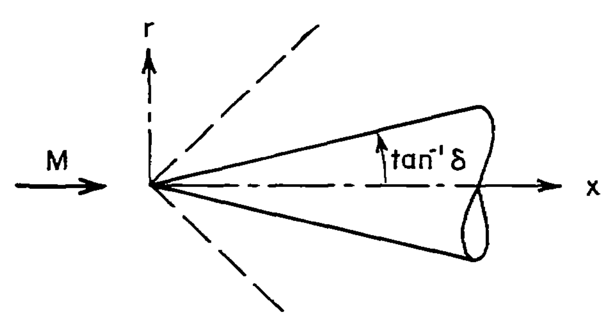
of the solution as a whole. This is true also of the first-order slender-body approximation. One can permit discontinuities in curvature and even slope, and pointed and even round subsonic ends, provided one attaches no significance to the result close to the resulting singularity in pressure, or subsequently corrects the solution locally by techniques that have been developed for discontinuities in slope in supersonic (ref. 18) and subsonic flow (ref. 19) and for subsonic ends (ref. 20). (Round supersonic noses can probably also be permitted, and could be corrected locally if the exact solution were known for supersonic flow past a paraboloid of revolution.)

In the second approximation of slender-body theory (just as in thin-wing theory) the restrictions become more severe, and the remedies correspondingly more complicated, and it is no longer always true that the formal solution is correct except locally. These difficulties are greater for subsonic flow because not only are the bodies of interest usually blunter (round noses being the rule), but also the restrictions are greater (pointed noses being excluded, whereas they are admitted in supersonic flow since no stagnation points appear).

Consequently, application of the present theory to examples of subsonic flow will be postponed until nose corrections have been discussed. To illustrate the theory, a few examples will now be given for supersonic flow. No difficulties appear if the ends are pointed, the meridian curve is elsewhere analytic, and one does not inquire too closely into the details of the flow near a pointed tail - where the flow is actually subsonic and, in any case, the real flow is determined by viscosity.

Cone

Consider a cone whose slope is δ (sketch (b)), so that the body is described by $R = \delta x$. With the origin of coordinates at the vertex $(a = 0)$, the slender-body potential of equations (9) is



Sketch (b).- Supersonic flow past cone.

$$\varphi(x,r) = \delta^2 x \left(\ln \frac{Br}{2x} + 1 \right) \tag{23a}$$

Then equation (12c) gives

$$f(x) = -\delta^4 x \left[(2M^2 - 1) \ln \frac{B\delta}{2} + M^2 N + M^2 + \frac{1}{2} \right] \tag{23b}$$

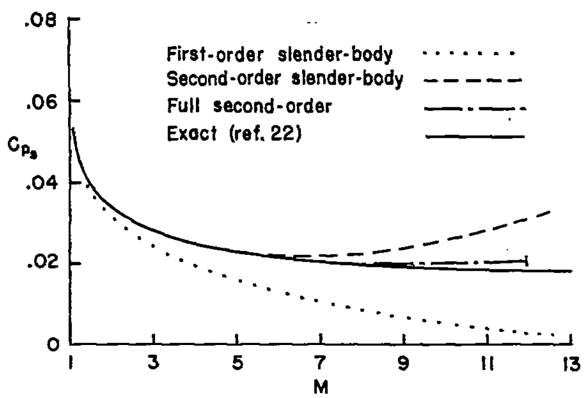
and equations (12) give as the second-order perturbation potential

$$\phi(x,r) = \delta^2 x\left(ln\,\frac{Br}{2x}+1\right)+\delta^4 x\left\{M^2 ln^2\,\frac{Br}{2x}-\left[(2M^2-1)ln\,\frac{B\delta}{2}+\frac{1}{2}\right]\left(ln\,\frac{Br}{x}+1\right)-\right.$$

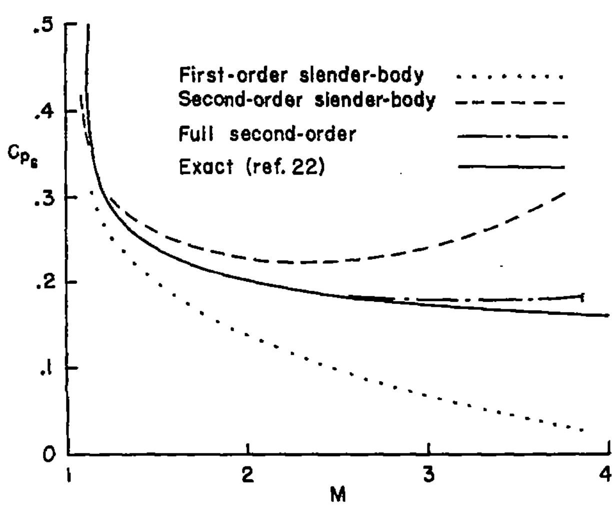
$$\left.M^2(N+1)-\frac{1}{4}\,B^2\,\frac{r^2}{\delta^2 x^2}-\frac{1}{4}\,M^2\,\frac{\delta^2 x^2}{r^2}\right\} \tag{23c}$$

The surface pressure coefficient is, from equation (13),

$$C_{p_s} = \delta^2\left(2\;ln\,\frac{2}{B\delta}-1\right)+\delta^4\left[3B^2 ln^2\,\frac{2}{B\delta}-(5M^2-1)ln\,\frac{2}{B\delta}+2M^2N+\frac{13}{4}\,M^2+\frac{1}{2}\right] \tag{23d}$$



Sketch (c).- Pressure on cone of 5° semivertex angle (γ = 1.405).



Sketch (d).- Pressure on cone of 15° semivertex angle (γ = 1.405).

which is the result first given by Broderick (ref. 8).

Broderick has compared the first- and second-order slender-body solutions with the exact results (ref. 22) for various cone angles. Two cases are reproduced in sketches (c) and (d), and the second approximation is seen to yield considerable improvement for moderate cone angles at speeds below the hypersonic range. Also shown are the results of the second-order theory which does not involve the further approximation of slender-body theory (ref. 15). The slender-body simplification is seen to reduce the numerical accuracy at high Mach numbers. The reason is that, roughly speaking, linearized theory and its second-order counterpart assume only that the thickness ratio τ is small, whereas the slender-body approximation implies also that $B\tau$ is small (ref. 23). The latter is a more serious restriction at Mach numbers appreciably in excess of $\sqrt{2}$. In the subsonic range, on the other hand, β cannot exceed 1, so that the slender-body simplification does not significantly reduce the numerical accuracy.

The corresponding result in the transonic small-disturbance approximation is found from equations (9) and (19), or simply by discarding all second-order terms in equations (23) except those involving N and setting $M = 1$. The result for surface pressure coefficient is, in the similarity form of equation (22),

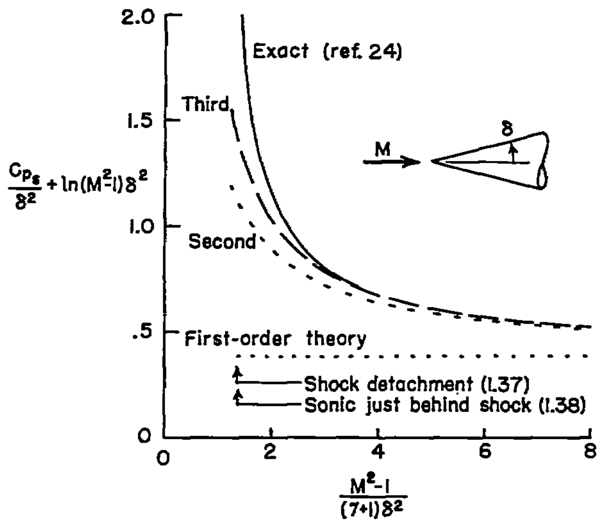
$$\frac{C_{p_s}}{\delta^2} + 2\ ln(B\delta) = P\left[\frac{B^2}{(\gamma+1)\delta^2}\right] = (2\ ln\ 2 - 1) + \frac{(\gamma+1)\delta^2}{B^2} \qquad (24a)$$

This series has been extended to a third approximation in unpublished work, giving

$$\frac{C_{p_s}}{\delta^2} + 2\ ln(B\delta) = (2\ ln\ 2 - 1) + \frac{(\gamma+1)\delta^2}{B^2} + \left(\frac{\pi^2}{12} - \frac{1}{4}\right)\left[\frac{(\gamma+1)\delta^2}{B^2}\right]^2 \qquad (24b)$$

Exact numerical solutions of the transonic small-disturbance problem have been calculated by Oswatitsch and Sjödin (ref. 24). The comparison of these results shown in sketch (e) gives an idea of the extent to which the present theory can penetrate into the transonic range.

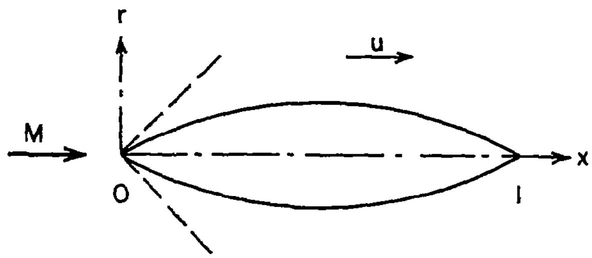
As indicated in sketch (e), detachment of the bow shock wave and attainment of sonic flow just behind the shock are both associated with a specific value of the transonic similarity parameter. However, this is not true (in contrast to plane flow) of the "upper critical Mach number" at which sonic flow is attained at the surface. This means that the limit of convergence of the small-disturbance series (such as eq. (24b)) cannot be associated with the first appearance of a subsonic zone in an otherwise supersonic flow field.



Sketch (e).- Correlation by transonic similarity rule of pressure on cone; $\gamma = 7/5$.

Parabolic Spindle

The analytic form of the second approximation grows complicated for shapes other than the cone, except in the further approximation of transonic small-disturbance theory.



Sketch (f).- Supersonic flow past parabolic spindle.

The analytic complexity of the full second approximation will be illustrated for the spindle formed by revolving a parabolic arc (sketch (f)). Let it be of unit length, with semivertex angle δ (so that its thickness ratio is $\delta/2$), and choose the origin of coordinates at the nose. Then the body is described by

$$R(x) = \delta x(1 - x) \tag{25}$$

The first-order slender-body solution of equations (9) is

$$\varphi = \delta^2 x\left[(1 - x)(1 - 2x)\ln\frac{Br}{2x} + 1 - \frac{9}{2}x + \frac{11}{3}x^2\right] \tag{26}$$

The second approximation is found using equations (12) together with the integrals of appendix B. Rather tedious computation gives as the surface pressure coefficient

$$\begin{aligned}
C_{p_s} = {} & \delta^2\left[2(1-6x+6x^2)\ln\frac{2}{B\delta(1-x)} - 1+16x-22x^2\right] + \\
& \delta^4\Bigg\{2\left[(1-2M^2)+6(7M^2-4)x(1-x)-30(5M^2-3)x^2(1-x)^2\right]\left[L_2(x)+\frac{1}{2}\ln^2(1-x)\right]+3B^2\left[1-20x(1-x)+72x^2(1-x)^2\right]\ln^2\frac{2}{B\delta(1-x)} + \\
& \left[(1-5M^2)+2(33M^2-24)x+(333-311M^2)x^2+2(271M^2-333)x^3+4(99-76M^2)x^4\right]\ln\frac{2}{B\delta(1-x)} + \\
& \left[6(7M^2-4)(1+2x)-5(5M^2-3)(1-25x-31x^2+25x^3)\right](1-x)\ln(1-x)+2M^2N\left[1-15x(1-x)+47x^2(1-x)^2\right] + \\
& \left(\frac{13}{4}M^2+\frac{1}{2}\right)+\left[12(7M^2-4)\ln\frac{2}{B\delta}+21-91M^2\right]x+\left[18(19-32M^2)\ln\frac{2}{B\delta}+\frac{3227}{6}M^2-\frac{515}{2}\right]x^2 + \\
& \left[220(5M^2-3)\ln\frac{2}{B\delta}+625-\frac{3047}{3}M^2\right]x^3+\left[125(3-5M^2)\ln\frac{2}{B\delta}+\frac{7081}{12}M^2-\frac{1661}{4}\right]x^4\Bigg\}
\end{aligned} \tag{27a}$$

Here $L_2(x)$ is Euler's dilogarithm, defined by

$$L_2(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2} = -\int_0^x \frac{\ln(1 - \xi)}{\xi}\,d\xi = \int_0^1 \frac{\ln\xi}{\xi - \frac{1}{x}}\,d\xi \tag{27b}$$

Keune and Oswatitsch (ref. 25) have encountered this function in their integral equation theory for slender bodies of revolution in transonic flow. They give a short table and references to further tables of which Powell's (ref. 26) is the most useful. In accord with the second-order similarity rule (ref. 27), the surface pressure coefficient has the general form

$$C_{p_s} = \tau^2 P(x;B\tau) + \tau^4\left[p_1(\ ) + M^2 p_2(\ ) + (\gamma + 1)\frac{M^4}{B^2}p_3(\ )\right] \tag{28a}$$

where τ is the thickness ratio. More specifically, it has the form appropriate to smooth slender bodies of revolution

$$C_{p_s} = \tau^2(P_1 ln\ \tau + P_2) + \tau^4\left[(p_{11} + M^2 p_{21}) ln^2\tau + (p_{12} + M^2 p_{22}) ln\ \tau + p_{13} + M^2 p_{23} + (\gamma + 1)\frac{M^2}{B^2} p_3\right] \tag{28b}$$

Enormous simplification results from the approximation of transonic small-disturbance theory. The second-order effect is then given by equation (19a) as

$$\Delta_2\phi = -\frac{\gamma + 1}{2}\frac{\delta^4}{M^2 - 1}\left(x - \frac{15}{2}x^2 + \frac{62}{3}x^3 - \frac{47}{2}x^4 + \frac{47}{5}x^5\right) \tag{29}$$

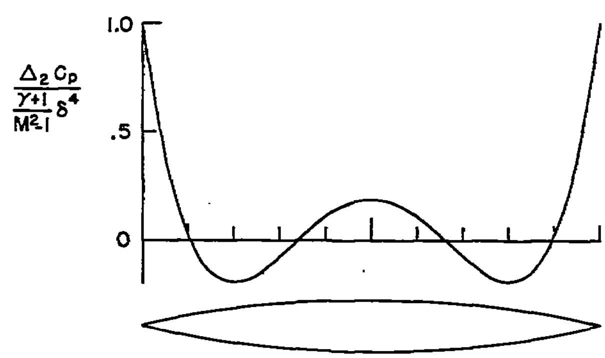
Hence second-order effects alter the pressure coefficient at any point by

$$\Delta_2 C_p = -2\Delta_2\phi_x = \frac{\gamma + 1}{M^2 - 1}\delta^4(1 - 15x + 62x^2 - 94x^3 + 47x^4) \tag{30}$$

which is plotted in sketch (g). Adding the first-order contribution gives, on the surface,

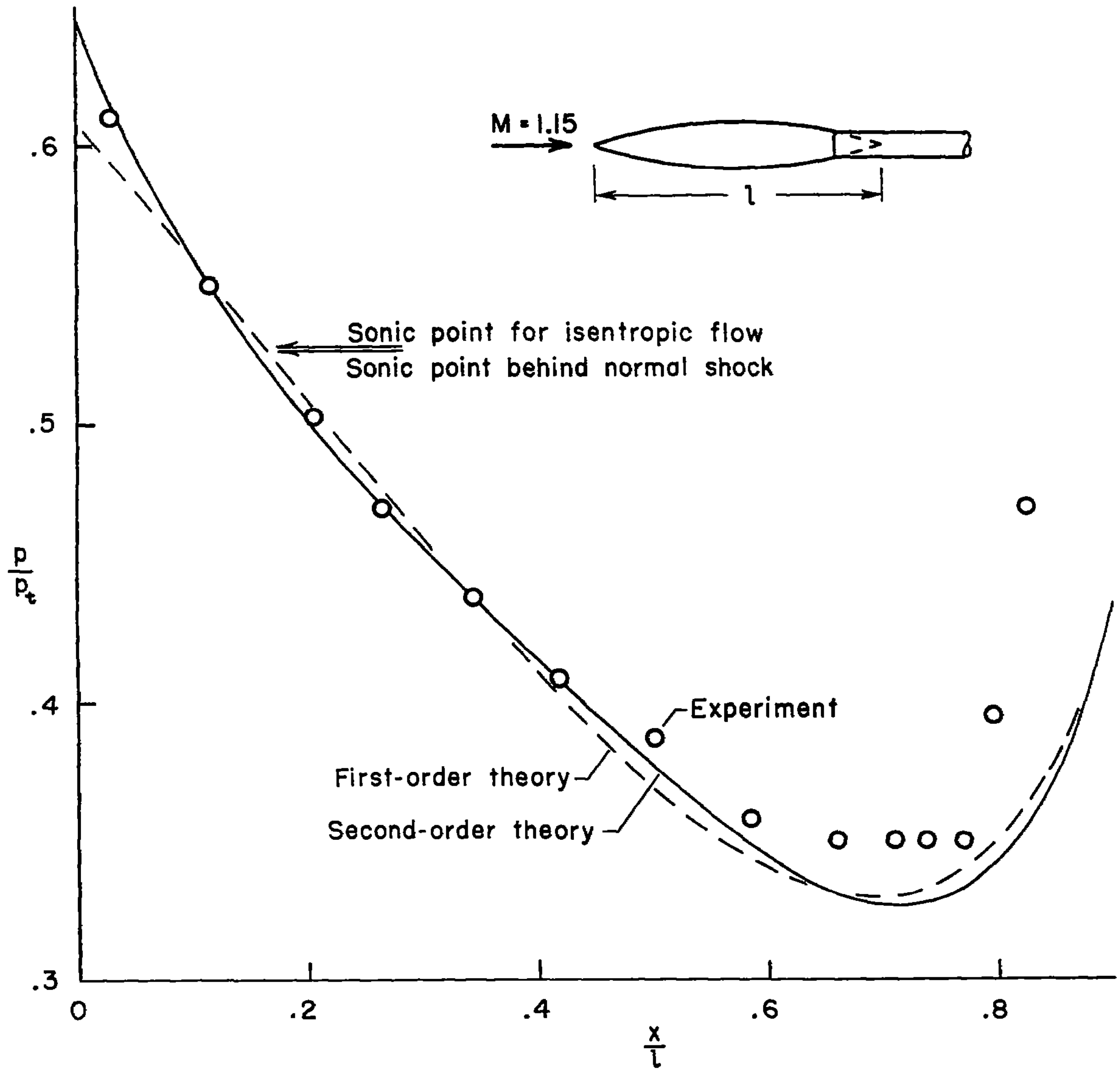
$$C_{p_s} = \delta^2\left[2(1 - 6x + 6x^2) ln\frac{2}{B\delta(1 - x)} - 1 + 16x - 22x^2\right] + \frac{\gamma + 1}{M^2 - 1}\left(\frac{\delta}{2}\right)^4\left[3 - 34(2x - 1)^2 + 47(2x - 1)^4\right] \tag{31}$$

Here the second-order term has been rewritten to make clear that it is symmetric about the middle of the body, as indicated in sketch (g), and so contributes nothing to the drag.



Sketch (g).- Second-order increment in pressure on spindle.

Sketch (h) compares this simple result with Drougge's measurements of pressure on a parabolic-arc spindle of thickness ratio 1/6 at $M = 1.15$ (ref. 28). It is remarkable that the first and second approximations give successively more accurate values in the region of subsonic flow, which is of considerable extent because the free-stream Mach number is somewhat below the value (1.18) for detachment of the bow wave.



Sketch (h).- Pressure on parabolic spindle with $\tau = 1/6$ at $M = 1.15$.

SUBSONIC FLOW

It has already been pointed out that only under rather severe restrictions on body smoothness will the second-order slender-body solution be uniformly valid over the entire surface of the body. In particular, it fails at least locally near stagnation points, and these can scarcely be avoided in practice for subsonic flows. We therefore consider to what extent the formal solution breaks down - and how it can be corrected - for subsonic flow past a body that has sharp (conical) or round, rather than cusped, ends, but elsewhere satisfies the smoothness requirements. (Violations of the restrictions elsewhere than at the ends - for example, at discontinuities in slope - could be treated by analogous methods; see refs. 18 and 19.)

Failure at Subsonic Ends

Just as in plane flow (ref. 20) it turns out that the formal second-order solution for a body with stagnation points may have one of three degrees of validity:

1. Valid except near stagnation points where it predicts infinite surface speeds

2. Invalid everywhere, but finite except near stagnation points

3. Infinite everywhere

These three cases are successively more serious (and are accordingly associated with successively greater blunting), except that the second is more insidious than the third because it gives no warning.

The distribution of these three cases with respect to nose bluntness and Mach number is compared in the following table with the corresponding results for airfoils. A regular trend is apparent, bodies being at least as critical as airfoils, with the one exception of sharp noses in subsonic flow. There, however, the difficulties of case 2 do arise in the various components of the solution but happen to cancel in the net result. (Furthermore, the corresponding airfoil problem could be put into case 1 by manipulating, by partial integration, the integrals involved in the second-order theory.)

Case	Bodies (using ϕ)	Airfoils	
		Using ϕ	Using ψ
1. Valid except near stagnation points	Sharp, $M = 0$ [1]Sharp, $M > 0$	Sharp, $M = 0$ Round, $M = 0$	Sharp, $M = 0$ Sharp, $M > 0$ Round, $M = 0$
2. Invalid everywhere, but finite except at stagnation points	Round, $M = 0$	Sharp, $M > 0$	Round, $M > 0$
3. Infinite everywhere	Round, $M > 0$	Round, $M > 0$	

[1]Except for this one case, placement in the first category has been definitely established by actual worked examples (using the Janzen-Rayleigh method). In this exceptional case, the placement is based instead on the absence of algebraic singularities, which might have to be modified by the source eigensolutions discussed below, from the second-order solution given below for the spindle. It would be well, however, to confirm this classification by carrying out the Janzen-Rayleigh solution for a conical tip.

In the first case, local failure occurs because the true speed is proportional to z^ϵ near a sharp nose and to $\sqrt{\frac{z}{z+\epsilon^2}}$ near a round nose (where z is the distance into the nose and ϵ is proportional to the body thickness), but the slender-body expansion forces these into the formal series

$$z^\epsilon = 1 + \epsilon^2 \ln z + O(\epsilon^4)$$

$$\sqrt{\frac{z}{z+\epsilon^2}} = 1 - \frac{1}{2}\frac{\epsilon^2}{z} + O(\epsilon^4)$$

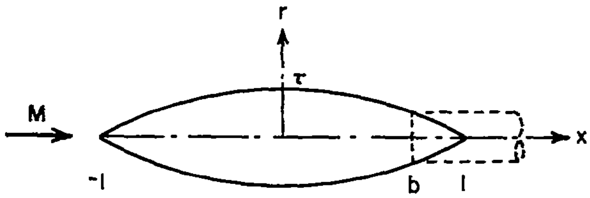
which are not uniformly valid near $z = 0$. Recognition of this source of the singularities permits one to formulate simple rules for rendering the formal solution uniformly valid, with the aid of the correct solution for some simple body having the same nose shape (refs. 20 and 13).

In the second case listed, the over-all failure results from singular eigensolutions - extraneous solutions that satisfy the second-order equation and the slender-body boundary conditions. They enter because of the inexactness of the slender-body tangency condition near the nose. The eigensolution is a point source located at the stagnation point if one works with the velocity potential (and a dipole if one works with the stream function). In plane flow there are at least three simple ways to exclude false eigensolutions, but unfortunately none of them is applicable in axisymmetric flow. First, the source eigensolution can be excluded in plane flow by working with the stream function (which imposes a condition on mass flow that would be violated by an extraneous source). However, Stokes' stream function, which should exclude source eigensolutions in the same way for axisymmetric flow, fails for other reasons to yield the correct second approximation (ref. 29). Second, the source or dipole eigensolution can simply be deleted as inadmissible in plane flow, and the remainder is the correct solution. In axisymmetric flow, however, the true slender-body solution may contain a term indistinguishable from an eigensolution. Third, there exists a similarity rule that relates surface quantities on a single plane airfoil in subsonic flow to those in the corresponding incompressible problem (ref. 13), which is free of eigensolutions. No such rule exists for bodies of revolution, however, nor does the difficulty disappear at zero Mach number. Indeed, it is only for round noses in incompressible flow that eigensolutions arise (see preceding table); and they can therefore be handled by comparison with the known solution for incompressible flow past a paraboloid of revolution.

In the third case listed, divergent integrals arise in the second approximation. They can be assigned a finite interpretation only by solving the problem by another approximation - either the Janzen-Rayleigh expansion in powers of M^2, or the full second-order theory without the slender-body approximation. The Janzen-Rayleigh solution is uniformly valid near the stagnation point, and the second-order slender-body solution can be extracted from it using the second-order similarity rule (ref. 27). The full second-order solution involves source eigensolutions, but they can be eliminated by requiring conservation of mass within a large contour that lies everywhere far from the region of nonuniformity at the nose. (This cannot be done with the slender-body solution because it is not valid far from the body.) Both these procedures have recently been carried out for the paraboloid of revolution, with identical results (ref. 14). It will be shown here how any other round-nosed body can be treated with the aid of that solution.

Sharp Ends - The Parabolic Spindle

Sharp-ended bodies in subsonic flow have stagnation points, but the formal second approximation, like the first, is correct except very near the tips (case 1 of the preceding table). It can be corrected even there by simple rules (ref. 20). However, the region is so minute (being of exponentially small order in the body thickness) that the correction is usually of no practical significance and will be ignored here.



Sketch (i).- Subsonic flow past parabolic spindle.

As an example, consider again the parabolic-arc spindle. Because symmetrical bodies in subsonic flow induce symmetric disturbances, it is convenient to choose the origin at the middle (sketch (i)). The spindle of thickness τ is described by

$$R(x) = \tau(1 - x^2) \tag{32}$$

The slender-body solution of equations (9) is

$$\varphi = \tau^2 x\left[2(1 - x^2)\, l n\, \frac{2\sqrt{1 - x^2}}{\beta r} - 3 + \frac{11}{3} x^2\right] \tag{33}$$

The second approximation is found from equations (12) and appendix B. The result for the streamwise velocity component on the surface is

$$\begin{aligned}
\frac{u}{U} = {} & 1+\tau^2(1-3x^2)\left(2\ l n\, \frac{2}{\beta\tau\sqrt{1-x^2}} - 3\right) + \\
& \tau^4\Bigg\{\beta^2(5-42x^2+45x^4)\left(\frac{\pi^2}{12} - \frac{1}{4}\ l n^2\ \frac{1+x}{1-x} + 2\ l n^2 \frac{2}{\beta\tau\sqrt{1-x^2}}\right) + \\
& 2M^2(1-12x^2+15x^4)\left(\frac{1}{4}\ l n^2\ \frac{1+x}{1-x} - \frac{\pi^2}{12}\right) + \left[15(3-5M^2)x^2 - (27-41M^2)\right] x\ l n\ \frac{1+x}{1-x} - \\
& \left[\left(\frac{55}{2} - \frac{61}{2} M^2\right) - (271-325M^2)x^2 + \left(\frac{615}{2} - \frac{797}{2} M^2\right)x^4\right] l n\, \frac{2}{\beta\tau\sqrt{1-x^2}} + \\
& \left(\frac{137}{8} - \frac{493}{24} M^2 + 3M^2 n\right) - \left(\frac{553}{4} - \frac{2413}{12} M^2 + 34M^2 n\right)x^2 + \\
& \left(\frac{1049}{8} - \frac{5629}{24} M^2 + 47M^2 n\right)x^4\Bigg\}
\end{aligned} \tag{34}$$

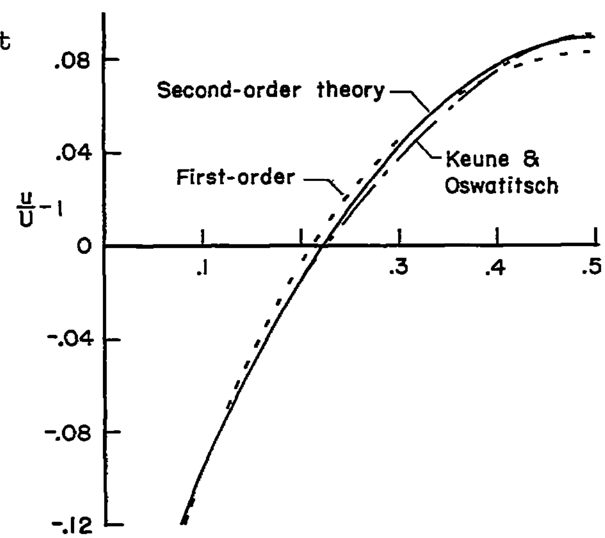
The pressure coefficient can be calculated from equation (4), and has again the form of equations (28). The maximum velocity, which occurs on the middle of the spindle, is given by

$$\begin{aligned}
\frac{u_{max}}{U} = {} & 1 + \tau^2\left(2\ l n\ \frac{2}{\beta\tau} - 3\right) + \tau^4\Bigg[10\beta^2 l n^2\ \frac{2}{\beta\tau} - \left(\frac{55}{2} - \frac{61}{2} M^2\right) l n\ \frac{2}{\beta\tau} + \\
& \frac{\pi^2}{12}\ (5 - 7M^2) + \frac{137}{8} - \frac{493}{24} M^2 + 3M^2 n\Bigg]
\end{aligned} \tag{35}$$

Again, enormous simplification results from the approximation of transonic small-disturbance theory. The second-order effect is then found from equations (19) to be the same in subsonic as in supersonic flow. It is therefore given by equations (29) and (30), with x replaced by $(1+x)/2$ because of the difference in coordinates, and ϵ by 2τ. Hence the surface pressure coefficient is given by

$$C_p = 2\tau^2\left[(3x^2-1)\left(2\ ln\frac{2}{\beta\tau\sqrt{1-x^2}}-3\right)-2x^2\right]-\frac{\gamma+1}{1-M^2}\tau^4(3-34x^2+47x^4) \tag{36}$$

The spindle in subsonic flow has been treated in the transonic small-disturbance approximation also by Keune and Oswatitsch (ref. 25), who solve an approximate integral equation numerically. Their result for the perturbation velocity on a 14.6-percent-thick spindle at M = 0.90 (which is nearly the critical Mach number) is shown in sketch (j) to compare reasonably well with the present result.[4] In particular, their curve crosses that of linearized theory twice on each half of the body, as the second-order solution does (sketch (g)).



Sketch (j).- Pressure on parabolic spindle with $\tau = 0.146$ at M = 0.90.

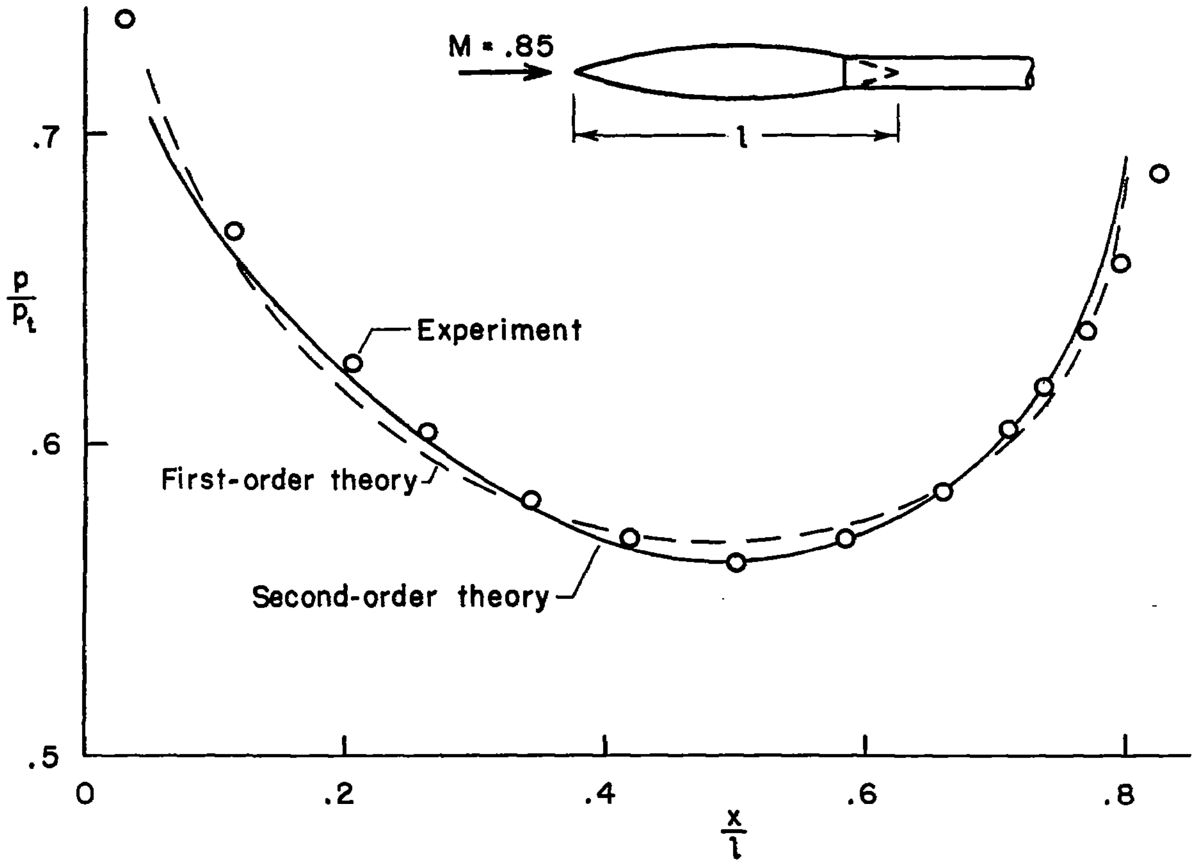
Drougge (ref. 28) has tested a parabolic spindle truncated by a support sting (cf. sketch (h)). If the base lies at $x = b$ (sketch (i)), the first-order slender-body solution gives

$$C_p = 2\tau^2\left[(3x^2-1)\ln\frac{4(b-x)}{\beta^2\tau^2(1+x)(1-x)^2}+\frac{3}{2}(1+b^2)-3(1-b)x-11x^2+b\,\frac{1-b^2}{b-x}\right] \tag{37}$$

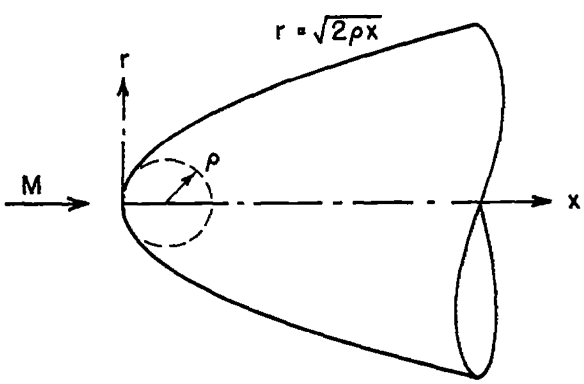
The algebraic singularity at the corner ($x = b$) should be corrected by the techniques of references 19 and 20. In the second approximation the corner introduces divergent integrals (just as a round nose does). This difficulty has been avoided by using the second-order increment for the complete spindle (the second term in eq. (36)), which should be a

[4]Keune and Oswatitsch solve equation (17) with $(\gamma+1)$ replaced by $M^*(1-M^2)/(1-M^*)$, where $M^{*2} = (\gamma+1)M^2/[2+(\gamma-1)M^2]$; this change has therefore been made also in equation (36) in calculating the curve in sketch (j).

satisfactory approximation away from the corner. The result is compared in sketch (k) with the measured pressures at $M = 0.85$, and the second-order terms are seen to improve the agreement.



Sketch (k).- Pressure on parabolic spindle with $\tau = 1/6$ at $M = 0.85$.



Sketch (l).- Notation for paraboloid of revolution.

Incompressible Flow Past Paraboloid

Consider now the case in which eigensolutions may invalidate the second approximation everywhere. According to the preceding table, this case (case 2) can occur only for round noses in incompressible flow. We consider, therefore, first the prototype of round-nosed bodies, a paraboloid of revolution. With the nose at the origin, it may be described by $y = \sqrt{2\rho x}$, where ρ is the nose radius (sketch (l)). Although the infinite paraboloid has properly no thickness ratio (or is an ellipsoid of zero thickness ratio),

$\sqrt{\rho}$ formally assumes that role. From equations (9) the first-order slender-body potential is found to be, aside from an irrelevant constant (which includes the "infinite constant" $-(\rho/2)\ln b$ of eq. (9c))

$$\varphi = \frac{1}{2}\rho \ln \frac{r^2}{x} \tag{38a}$$

Then from equations (12) the formal second approximation is found to be

$$\phi \overset{?}{=} \frac{1}{2}\rho \ln \frac{r^2}{x} - \frac{1}{8}\rho \frac{r^2}{x^2} \tag{38b}$$

Here the second-order term is actually incorrect. The exact perturbation potential for the paraboloid in incompressible flow is known (e.g., from separation of variables in parabolic coordinates) to be

$$\left.\begin{aligned}\Phi &= \frac{1}{2}\rho \ln 2\left[\sqrt{\left(x - \frac{1}{2}\rho\right)^2 + r^2} - \left(x - \frac{1}{2}\rho\right)\right] \\ &= \frac{1}{2}\rho \ln \frac{r^2}{x} + \frac{1}{8}\rho\left(\frac{2\rho}{x} - \frac{r^2}{x^2}\right) + O(\rho^3,\ \rho^2 r^2,\ \rho r^4)\end{aligned}\right\} \tag{39}$$

Thus the formal solution of equation (38b) is seen to be in error by a term $\rho^2/4x$, which affects the pressures everywhere.

This term is an eigensolution for the slender-body problem, because it satisfies trivially the equation $\varphi_{rr} + \varphi_r/r = 0$ without affecting the slender-body tangency condition of equation (3a). Moreover, it has the proper behavior at infinity, because it is in fact the slender-body representation of a point source located at (or within a distance of order ρ of) the origin. Thus the exact perturbation potential for a point source of strength $\rho^2/4$ located on the axis at $x = k\rho$ is

$$\Phi = \frac{1}{4}\rho^2 \frac{1}{\sqrt{(x - k\rho)^2 + r^2}} = \frac{1}{4}\frac{\rho^2}{x} + O(\rho^3,\ \rho^2 r^2)$$

Alternatively, the eigensolution may be regarded as representing a second-order uncertainty in the location of the nose. For replacing x in equation (38b) by $x - (\rho/2)$ yields the correct result of equation (39).

Eigensolutions at Round Ends in Incompressible Flow

The extraneous eigensolution arises in the formal solution for the paraboloid because of the inexactness of the tangency condition near the nose; consequently, just the same error will arise for any other body having a round nose of the same radius. That is, the formal second-order slender-body potential will be too small by an amount

$$\Delta g = \frac{\rho_a^2}{4(x - a)}$$

where ρ_a is the radius of the nose, located at $x = a$. A corresponding error will arise at a round rear end, where the eigensolution is the slender-body representation of a sink rather than a source. Hence, the formal solution of equations (12) can be corrected by calculating $g(x)$ not from equation (12d) but from

$$g(x) = f(x) \ln \frac{1}{2\sqrt{(x - a)(b - x)}} + \frac{1}{2} \int_a^b \frac{f(x) - f(\xi)}{|x - \xi|} d\xi + \frac{1}{4}\left(\frac{\rho_a^2}{x - a} - \frac{\rho_b^2}{b - x}\right) \tag{40}$$

This modification gives a solution that is valid to second order except within a distance of the order of the radius from either round end, where singularities remain. That is, removal of the spurious eigensolution by means of equation (40) reduces the difficulty from case 2 of the preceding table to case 1. For example, the surface speed on a paraboloid of revolution is found, either from equation (39) or simply from Munk's rule (ref. 30) that the speed on any ellipsoid subjected to incompressible flow along an axis is the projection of the maximum velocity, to be

$$\frac{q}{U} = \left[\frac{x}{x + (\rho/2)}\right]^{1/2} \tag{41a}$$

Expanding this formally for small ρ yields

$$\frac{"q"}{U} = 1 - \frac{1}{4}\frac{\rho}{x} + \frac{3}{32}\frac{\rho^2}{x^2} - \ldots \tag{41b}$$

and this is also the result of the present theory, the first two terms being the usual slender-body result, and the third the second-order increment after removal of spurious eigensolutions. The remaining singularities are such that even in first-order theory the integral for drag calculated from surface pressure is divergent (though this is not a serious difficulty because the drag is known to be zero).

Rules for Rendering Solution Valid Near Round Ends in Incompressible Flow

The singularities remaining at a round nose can be eliminated, and a uniformly valid approximation obtained by applying simple rules to the formal solution. Derivation of these rules requires a knowledge of the exact solution for some body that matches the one under consideration near its nose. The paraboloid of revolution is the prototype of round-nosed axisymmetric bodies. It was shown in reference 20 that the ratio of the exact solution for the paraboloid (eq. (41a)) to its formal series expansion (eq. (41b)) serves as a multiplicative correction factor for any round-nosed body. This rule renders the solution correct to second order[5] for uncambered airfoils (to which it also applies), but only to first order for bodies of revolution.

A second-order rule for bodies was derived (ref. 20) by considering the exact solution for an ellipsoid (or hyperboloid), which matches the nose more closely than does a paraboloid. It should be pointed out that in this case one cannot simply use the ratio of the exact solution to its formal expansion because this would introduce a spurious stagnation point at the remote end of the ellipsoid. What one actually requires is the exact solution for a semi-infinite body that matches the nose to the required order, and this can be extracted from the solution for the ellipsoid.

The result is that for a body of revolution having a round nose at $x = 0$, described by

$$r^2 = R^2(x) = 2\rho x - Bx^2 + \ldots \tag{42a}$$

the formal second-order slender-body solution "q_2" for surface speed is converted into a uniformly valid second approximation $\overline{q}_2$ by the rule

$$\frac{\overline{q}_2}{U} = \frac{1}{\sqrt{1+\lambda}}\left(\frac{"q_2"}{U} + \frac{\lambda}{2}\frac{"q_1"}{U} - \frac{1}{8}\lambda^2\right), \qquad \lambda = \frac{\rho}{2x} - \frac{3}{4}B \tag{42b}$$

where "q_1" is the first-order part of "q_2".

A body with two round ends can be treated by applying this rule twice, shifting coordinates so that in equations (42) x is always measured into the end. The result can be simplified somewhat to the following. For a body having round nose and tail of radii ρ_a and ρ_b located at $x = a$ and $x = b$:

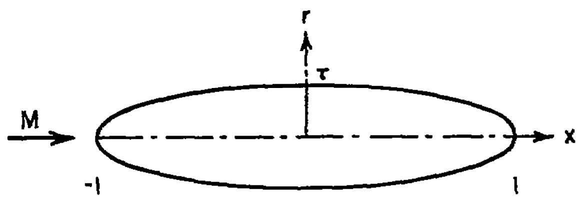
$$\frac{\overline{q}_2}{U} = \frac{1}{\sqrt{(1+\lambda_a)(1+\lambda_b)}}\left[\frac{"q_2"}{U} + \frac{1}{2}(\lambda_a+\lambda_b)\frac{"q_1"}{U} - \frac{1}{8}(\lambda_a-\lambda_b)^2\right], \tag{42c}$$

$$\lambda_a = \frac{\rho_a}{2(x-a)} - \frac{3}{4}B_a, \qquad \lambda_b = \frac{\rho_b}{2(b-x)} - \frac{3}{4}B_b$$

Corresponding rules for treating surface pressure directly have been given for airfoils (ref. 13), and could readily be deduced also for bodies.

[5]As pointed out in reference 20, the order of terms is counted in such a way that disturbances in velocity or pressure are always of first order. Thus a first-order term is only of $O(\tau^2 \ln \tau)$ near the middle of a slender body, but is $O(1)$ near a stagnation point.

Rules for treating sharp ends have been given in reference 20; but, as discussed in connection with the spindle, the region of nonuniformity is so small as to be of no practical significance.



Sketch (m).- Ellipsoid of revolution in subsonic flow.

Example: Incompressible Flow Past Ellipsoid

Consider a slender ellipsoid of revolution in incompressible flow. With axes chosen as shown in sketch (m), the ellipsoid of thickness ratio τ is described by

$$r = R(x) = \tau\sqrt{1-x^2} \tag{43}$$

Equations (9) and (12) give

$$\left.\begin{aligned} F(x) &= -\tau^2 x \\ G(x) &= \tau^2 x\left(ln\, 2\sqrt{1-x^2} - 1\right) \\ f(x) &= -\tau^4\left(ln\, \frac{2}{\tau} + \frac{1}{2}\right)x \end{aligned}\right\} \tag{44a}$$

The radii of the nose and tail are $\rho_a = \rho_b = \tau^2$, so that equation (40) gives

$$k(x) = \tau^4\left(ln\, \frac{2}{\tau} + \frac{1}{2}\right)x\left(ln\, 2\sqrt{1-x^2} - 1\right) - \frac{1}{2}\,\tau^4\,\frac{x}{1-x^2} \tag{44b}$$

Then from equation (12a) the second-order slender-body solution for the perturbation potential is

$$\phi = \tau^2\left[1 + \tau^2\left(ln\, \frac{2}{\tau} + \frac{1}{2}\right)\right]x\left(ln\, \frac{2\sqrt{1-x^2}}{r} - 1\right) + \frac{1}{4}\,\tau^2 r^2 x\,\frac{3-x^2}{(1-x^2)^2} - \frac{1}{2}\,\tau^4\,\frac{x}{1-x^2} \tag{44c}$$

It can be verified that this is the asymptotic expansion, to this order, of the known exact solution for flow past an ellipsoid.

The streamwise velocity component and resultant speed on the surface are found to be

$$\frac{u_2}{U} = 1 + \tau^2\left(\ln\frac{2}{\tau} - \frac{1}{1 - x^2}\right) + \tau^4\left[\ln^2\frac{2}{\tau} - \frac{1}{2}\,\frac{1 + x^2}{1 - x^2}\ln\frac{2}{\tau} - \frac{1}{4} + \frac{x^2}{(1 - x^2)^2}\right] \tag{45a}$$

$$\frac{q_2}{U} = 1 + \tau^2\left[\ln\frac{2}{\tau} - \frac{1}{2}\left(1 + \frac{1}{1 - x^2}\right)\right] + \tau^4\left[\ln^2\frac{2}{\tau} - \frac{1}{2}\,\frac{1}{1 - x^2}\ln\frac{2}{\tau} - \frac{3}{8} - \frac{1}{4}\,\frac{1}{1 - x^2} + \frac{3}{8}\,\frac{1}{(1 - x^2)^2}\right] \tag{45b}$$

Applying the rule of equation (42c), with $\rho_a = \rho_b = B_a = B_b = \tau^2$, gives the uniformly valid result for surface speed (or pressure coefficient)

$$\frac{\bar{q}_2}{U} = \sqrt{1 - \bar{C}_{p_2}} = \frac{1 + \tau^2\left(\ln\frac{2}{\tau} - \frac{5}{4}\right) + \tau^4\ln\frac{2}{\tau}\left(\ln\frac{2}{\tau} - \frac{3}{4}\right)}{\left[1 + \tau^2\left(\frac{1}{1 - x^2} - \frac{3}{2}\right) + \frac{1}{2}\tau^4\left(\frac{9}{8} - \frac{1}{1 - x^2}\right)\right]^{1/2}} \tag{46a}$$

The exact result is

$$\frac{q}{U} = \frac{1 + \tau^2\,\dfrac{\operatorname{sech}^{-1}\tau - \sqrt{1 - \tau^2}}{\sqrt{1 - \tau^2} - \tau^2\operatorname{sech}^{-1}\tau}}{\left(1 + \tau^2\,\dfrac{x^2}{1 - x^2}\right)^{1/2}} \tag{46b}$$

As an extreme test, the approximate and exact values are compared in the following table for an ellipsoid of thickness ratio $\tau = 1/3$:

$\pm x$	0	0.4	0.8	0.9	0.95	0.98	1
q/U	1.122	1.110	1.025	0.924	0.788	0.584	0
$\bar{q}_2/U$	1.114	1.103	1.018	.918	.782	.580	0

Subsonic Flow Past Paraboloid

The remaining case to be disposed of is that of subsonic flow past a round nose. This is case 3 of the table on page 18, in which the formal second approximation leads to divergent integrals. This will be illustrated for the paraboloid (sketch (l)); and comparison with the known correct solution will again provide appropriate corrections.

The first-order slender-body solution for the paraboloid is found to be independent of Mach number, so that it is given by equation (38a). However, the potential is indeterminate to within an additive constant which was dropped there but must now be retained for purposes of the comparison. Hence the slender-body potential is written to include an arbitrary constant K as

$$\varphi = \rho\left(\frac{1}{2}\ln\frac{r^2}{x} + K\right) \tag{47}$$

In the second approximation, equation (12c) gives as the increment in source strength

$$f(x) = -\frac{1}{4}M^2\rho^2\frac{1}{x} \tag{48a}$$

and difficulties appear because this is not integrable at the nose. The function $g(x)$ of equation (12d) may be written formally as

$$g(x) = \frac{1}{8}M^2\frac{\rho^2}{x}\left(\ln\frac{4x^3}{\beta^2\epsilon} + \int_0^{\epsilon}\frac{d\xi}{\xi}\right) \tag{48b}$$

If ϵ is here regarded as small, all difficulties have been concentrated into an integral over a short portion of the nose. The integral diverges so that it is meaningless as it stands, nor can any a priori significance be assigned to it as a finite part. The proper interpretation is rather to be found from comparison with the known solution.

The formal second approximation of equation (12a) thus becomes

$$\begin{aligned}\phi = {} & \rho\left(K - \frac{1}{2}\ln\frac{x}{r^2}\right) - \frac{1}{8}\beta^2\rho\frac{r^2}{x^2} + {} \\ & \frac{1}{4}M^2\rho^2\left[\frac{1}{x}\left(\frac{1}{2}\ln\frac{x}{r^2} + \frac{1}{2}\ln\frac{4x^4}{\beta^2\epsilon r^4} + 2n - 2K + \frac{1}{2}\int_0^{\epsilon}\frac{d\xi}{\xi}\right) - \frac{\rho}{r^2}\right]\end{aligned} \tag{48c}$$

whereas the correct result has been shown to be (ref. 14), aside from an irrelevant constant,

$$\phi = -\frac{1}{2}\rho\ln\frac{2\rho x}{r^2} + \frac{1}{8}\beta^2\rho\left(\frac{2\rho}{x} - \frac{r^2}{x^2}\right) + \frac{1}{4}M^2\rho^2\left[\frac{1}{x}\left(\frac{1}{2}\ln\frac{2\rho x}{r^2} + 2\ln\frac{2x}{\beta r} + n\right) - \frac{\rho}{r^2}\right] \tag{49}$$

These two expressions agree if the divergent integral is interpreted according to

$$\int_0^{\epsilon} \frac{d\xi}{\xi} = 2\left(\frac{\beta^2}{M^2} - n\right) + ln\ \frac{8\rho\epsilon}{\beta^2} + 4K \tag{50}$$

Eigensolutions at Round Ends in Subsonic Flow

Consider now the general case of a body having a round nose of radius ρ_a at $x = a$ and a round tail of radius ρ_b at $x = b$. The second-order increment to source strength $f(x)$ will consist of a regular function $f_*(x)$ plus the singular terms

$$\tilde{f}(x) = \frac{1}{4} M^2\left(\frac{\rho_b^2}{b - x} - \frac{\rho_a^2}{x - a}\right) \tag{51}$$

These give a divergent integral $\int_0^{\epsilon} d\xi/\xi$ at either end of the body. Each of these integrals can be interpreted according to equation (50) in terms of the corresponding radius ρ_a or ρ_b and a constant K_a or K_b that can be determined from the first-order solution. The result is that in place of equation (12d), the function $g(x)$ is given by

$$\begin{aligned}g(x) = {} & f(x) ln\ \frac{\beta}{2\sqrt{(x - a)(b - x)}} + \frac{1}{2}\int_a^b \frac{f_*(x) - f_*(\xi)}{|x - \xi|}\, d\xi + {} \\ & \frac{1}{8} M^2\left\{\frac{\rho_a^2}{x - a}\left[2\left(\frac{\beta^2}{M^2} - n\right) + ln\ \frac{8\rho_a(x - a)^2}{\beta^2(b - a)} + 4K_a\right] - {}\right. \\ & \left.\frac{\rho_b^2}{b - x}\left[2\left(\frac{\beta^2}{M^2} - n\right) + ln\ \frac{8\rho_b(b - x)^2}{\beta^2(b - a)} + 4K_b\right]\right\}\end{aligned} \tag{52a}$$

where

$$f_*(x) = f(x) - \frac{1}{4} M^2\left(\frac{\rho_b^2}{b - x} - \frac{\rho_a^2}{x - a}\right) \tag{52b}$$

$$K_a = \lim_{x \to a}\left[\frac{1}{\rho_a} G(x) + \frac{1}{2}\ ln(x - a)\right] \tag{52c}$$

$$K_b = \lim_{x \to b}\left[-\frac{1}{\rho_b} G(x) + \frac{1}{2}\ ln(b - x)\right] \tag{52d}$$

As M tends to zero this reduces to equation (40).

As in incompressible flow, this modification renders the solution valid to second order except within a distance of the order of the radius from either end. (Case 3 of the table on page 18 has been reduced to case 1.) The surface speed again contains singularities like $(x-a)^{-1}$ in the first-order terms and $(x-a)^{-2}$ in the second-order terms (cf. eq. (41b)). These can be eliminated, and the solution rendered uniformly valid, by a rule corresponding to equations (42). Derivation of the second-order form of this rule makes use of the formal solution for the ellipsoid, which must therefore be found first.

Example: Subsonic Flow Past Ellipsoid

Consider subsonic flow past the slender ellipsoid of revolution of sketch (m). According to equations (9), the first-order source strength F(x) is unchanged from the incompressible value of equation (44a), and G(x) is modified only by insertion of a factor β, so that

$$\left.\begin{aligned} F(x) &= -\tau^2 x \\ G(x) &= \tau^2 x\left(\ln\frac{2\sqrt{1-x^2}}{\beta}-1\right) \end{aligned}\right\} \tag{53a}$$

The coefficients of the constant terms in G(x) at the ends are, from equations (52c) and (52d),

$$K_a = K_b = 1+\ln\frac{\beta}{2\sqrt{2}} \tag{53b}$$

Equation (12c) gives as the formal second-order increment in source strength

$$f(x) = \tau^4 x\left[(2M^2-1)\ln\frac{2}{\beta\tau}+M^2(n-1)-\frac{1}{2}\right]+\frac{1}{2}M^2\tau^4\frac{x}{1-x^2} \tag{53c}$$

The quantity in brackets is the $f_*(x)$ required in equation (52a), and the remainder is the singular terms of equations (51) and (52b) that lead to divergent integrals. The integral in equation (52a) is trivial (being a multiple of the first-order one), with the result that

$$\begin{aligned} g(x) = &-\tau^4 x\left[(2M^2-1)\ln\frac{2}{\beta\tau}+M^2(n-1)-\frac{1}{2}\right]\left(\ln\frac{2\sqrt{1-x^2}}{\beta}-1\right)-\frac{1}{2}\beta^2\tau^4\frac{x}{1-x^2}- \\ &\frac{1}{2}M^2\tau^4\left(\ln\frac{\tau\sqrt{1-x^2}}{2}+2-n\right)\frac{x}{1-x^2}+\frac{1}{4}M^2\tau^4\left[\frac{\ln(1+x)}{1+x}-\frac{\ln(1-x)}{1-x}\right] \end{aligned} \tag{53d}$$

5M

Finally, equation (12a) gives as the second-order slender-body perturbation potential

$$
\begin{aligned}
\phi = {} & \tau^2\left\{1-\tau^2\left[(2M^2-1)\ln\frac{2}{\beta\tau}+M^2(n-1)-\frac{1}{2}\right]\right\}x\left(\ln\frac{2\sqrt{1-x^2}}{\beta r}-1\right)+\frac{1}{4}\beta^2\tau^2 x r^2\frac{3-x^2}{(1-x^2)^2}- \\
& \frac{1}{2}\beta^2\tau^4\frac{x}{1-x^2}+M^2\tau^4 x\left[\left(\ln\frac{2\sqrt{1-x^2}}{\beta r}-\frac{1}{1-x^2}\right)\left(\ln\frac{2\sqrt{1-x^2}}{\beta r}+n-1\right)+\frac{1}{4}\tau^2\frac{x^2}{r^2}\right]- \\
& \frac{1}{2}M^2\tau^4\left(\ln\frac{\tau\sqrt{1-x^2}}{2r}+2-n\right)\frac{x}{1-x^2}+\frac{1}{4}M^2\tau^4\left[\frac{\ln(1+x)}{1+x}-\frac{\ln(1-x)}{1-x}\right]
\end{aligned}
\tag{53e}
$$

As a partial check, it may be noted that this result conforms to the second-order similarity rule (ref. 27). Differentiating gives the velocity components. On the surface of the ellipsoid

$$
\begin{aligned}
\frac{u}{U} = {} & 1+\tau^2\left(\ln\frac{2}{\beta\tau}-\frac{1}{1-x^2}\right)+\tau^4\left\{\beta^2\ln^2\frac{2}{\beta\tau}+\left[M^2-M^2x^2\frac{3-x^2}{(1-x^2)^2}-\frac{1}{2}\frac{1+x^2}{1-x^2}\right]\ln\frac{2}{\beta\tau}+\right. \\
& M^2\left[\frac{1}{2}\frac{1+x^2}{(1-x^2)^2}\ln 2-\frac{\ln(1+x)}{4(1+x)^2}-\frac{\ln(1-x)}{4(1-x)^2}-\frac{3}{4}\frac{1-5x^2+2x^4}{(1-x^2)^2}\right]-\frac{1}{4}+\frac{x^2}{(1-x^2)^2}+ \\
& \left.\frac{1}{2}M^2 n\frac{1-5x^2+2x^4}{(1-x^2)^2}\right\}
\end{aligned}
\tag{54a}
$$

$$
\begin{aligned}
\frac{q}{U} = {} & 1+\tau^2\left(\ln\frac{2}{\beta\tau}-\frac{1}{2}\frac{1}{1-x^2}-\frac{1}{2}\right)+\tau^4\left\{\beta^2\ln^2\frac{2}{\beta\tau}+\left[M^2-M^2x^2\frac{3-x^2}{(1-x^2)^2}-\frac{1}{2}\frac{1}{1-x^2}\right]\ln\frac{2}{\beta\tau}+\right. \\
& M^2\left[\frac{1}{2}\frac{1+x^2}{(1-x^2)^2}\ln 2-\frac{\ln(1+x)}{4(1+x)^2}-\frac{\ln(1-x)}{4(1-x)^2}-\frac{3}{4}\frac{1-5x^2+2x^4}{(1-x^2)^2}\right]-\frac{3}{8}-\frac{1}{4}\frac{1}{1-x^2}+ \\
& \left.\frac{3}{8}\frac{1}{(1-x^2)^2}+\frac{1}{2}M^2 n\frac{1-5x^2+2x^4}{(1-x^2)^2}\right\}
\end{aligned}
\tag{54b}
$$

The maximum speed in the flow field (aside from spurious singularities at the ends of the ellipsoid that are to be removed) occurs at the surface in the middle, and is

$$
\frac{q_{max}}{U} = 1+\tau^2\left(\ln\frac{2}{\beta\tau}-1\right)+\tau^4\left[\beta^2\ln^2\frac{2}{\beta\tau}+\left(M^2-\frac{1}{2}\right)\ln\frac{2}{\beta\tau}+\frac{1}{2}M^2 n-\frac{1}{4}-M^2\left(\frac{3}{4}-\frac{1}{2}\ln 2\right)\right]
\tag{55}
$$

This agrees with the result of Schmieden and Kawalki (ref. 31) except for the coefficient of $M^2\tau^4$, which they give as $-5/4$ instead of $-\left(\frac{3}{4}-\frac{1}{2}\ln 2\right)$. They work with the perturbation form of Stokes' stream function rather than the velocity potential, which facilitates imposing the condition of tangent flow at the body (particularly since they impose the condition exactly, and only later extract the slender-body series). However, they retain only linear and quadratic terms in the equation of motion. The cubic term $M^2\varphi_r{}^2\varphi_{rr}$ in equation (11) yields a second-order effect, and the same is true of some quartic as well as cubic terms when one works with Stokes' stream function. Furthermore, the linearized equation for the stream function is not correct to first order except in the slender-body approximation and in any case does not form a proper basis for iterating to find the second approximation (ref. 29). Thus for supersonic flow past a circular cone, Schmieden and Kawalki's procedure was found to yield the second-order slender-body solution correct except for the term in $M^2\tau^4$. This is presumably true in general, so that the disagreement in that term found here for the ellipsoid might have been anticipated. The present solution predicts a maximum speed slightly higher than Schmieden and Kawalki's, which does not appear unreasonable in view of their comparison with the Janzen-Rayleigh solution to order M^6 for a sphere, in which their speed was somewhat low.

In the transonic small-disturbance approximation, the surface pressure coefficient is given simply by

$$C_{p_s} = -\tau^2\left(2\ln\frac{2}{\beta\tau} - 1 - \frac{1}{1-x^2}\right) - \frac{\gamma+1}{2}\,\frac{\tau^4}{1-M^2}\,\frac{1-5x^2+2x^4}{(1-x^2)^2} \tag{56}$$

Rules for Rendering Solution Valid Near Round Ends in Subsonic Flow

As in incompressible flow, the ratio of the exact solution for a paraboloid to its formal series expansion serves as a first-order multiplicative correction factor. Thus it has been shown (ref. 14) that the slender-body solution for surface speed is converted into a uniformly valid first approximation by the rule

$$\frac{\overline{q}_1}{U} = Q\left(\frac{2x}{\rho}\,,\ M\right)\left(\frac{"q_1"}{U} + \frac{\rho}{4x}\right) \tag{57}$$

where $Q(2x/\rho\,, M)$ is the speed ratio on a paraboloid of revolution of nose radius ρ at Mach number M. Although Q has not been found exactly, it can in principle be determined to any desired degree of accuracy by the Janzen-Rayleigh method. In reference 14 this has been done including terms in M^2, and numerical values have been tabulated.

Singularities remain if this first-order rule is applied to the second-order solution. A second-order rule is required; and its form can be deduced with the aid of the preceding solution for the ellipsoid.

Replacing x by x/c in equations (53) to (56) gives the solution for an ellipsoid whose length is $2c$ rather than 2 (cf. sketch (m)). Then for small values of the distance $z = x + c$ measured from the nose, the surface speed is found from equation (54b) to have the form

$$\frac{"q_2"}{U} = 1 + \tau^2\left[-\frac{c}{4z} + \left(\ln\frac{2}{\beta\tau} - \frac{5}{8}\right) + \ldots\right] + \tau^4\left[\frac{M^2}{4}\frac{c^2}{z^2}\ln\frac{\beta^2\tau^2 c}{2z} + \left(\frac{3}{32} + \frac{3}{8}M^2 - \frac{1}{4}M^2 n\right)\frac{c^2}{z^2} - \left(\frac{1}{4}\ln\frac{2}{\beta\tau} + \frac{1}{32}\right)\frac{c}{z} + \ldots\right] \tag{57a}$$

The parameters ρ and B of equation (42a) are related to the present c and τ by $\rho = \tau^2 c$ and $B = \tau^2$, and in those terms the above expression becomes

$$\frac{"q_2"}{U} = 1 + \left[-\frac{\rho}{4z} + \left(\frac{1}{2}\ln\frac{4}{\beta^2 B} - \frac{5}{8}\right)B + \ldots\right] + \left[\frac{M^2}{4}\frac{\rho^2}{z^2}\ln\frac{\beta^2\rho}{2z} + \left(\frac{3}{32} + \frac{3}{8}M^2 - \frac{1}{4}M^2 n\right)\frac{\rho^2}{z^2} - \left(\frac{1}{8}\ln\frac{4}{\beta^2 B} + \frac{1}{32}\right)\frac{B\rho}{z} + \ldots\right] \tag{57b}$$

This expansion could be used to form a second-order rule, but the result is simplified by first determining the corresponding expansion for a general body, and then choosing a simple special case.

It is clear that, corresponding to equation (57b), the solution for any round-nosed body described by equation (42a) will, near its nose, have the form

$$\frac{"q_2"}{U} = 1 + \left[-\frac{\rho}{4z} + R_1(z)\right] + \left[\frac{M^2}{4}\frac{\rho^2}{z^2}\ln\frac{\beta^2\rho}{2z} + \left(\frac{3}{32} + \frac{3}{8}M^2 - \frac{1}{4}M^2 n\right)\frac{\rho^2}{z^2} - C\frac{\rho}{z} + R_2(z)\right] \tag{58a}$$

Here C is a constant, and R_1 and R_2 are regular functions of z between which a relation will now be found. Although equations (57b) and (58a) are both singular at $z = 0$, their ratio must be regular. Dividing the latter by the former and expanding gives

$$1+\left[R_1(z)+\left(\frac{5}{8}-\frac{1}{2}\ln\frac{4}{\beta^2 B}\right)B+\ldots\right]+\left\{\left[\frac{1}{4}R_1(z)-C+\frac{3}{16}B\right]\frac{\rho}{z}+\ldots\right\}$$

and this is regular at $z = 0$ only if

$$C = \frac{3}{16}B + \frac{1}{4}R_1(0) \tag{58b}$$

There is a particular body for which equation (58) has the simplest form, namely that for which $R_1(z) = R_2(z) = 0$, so that

$$\frac{"q_2"}{U} = 1 - \frac{\rho}{4z} + \left[\frac{M^2}{4}\frac{\rho^2}{z^2}\ln\frac{\beta^2\rho}{2z} + \left(\frac{3}{32}+\frac{3}{8}M^2 - \frac{1}{4}M^2 n\right)\frac{\rho^2}{z^2} - \frac{3}{16}B\frac{\rho}{z}\right] \tag{59}$$

Note that except for the term in B this is just the second-order slender-body solution for the paraboloid.

Now suppose that the exact solution is known for any semi-infinite body having prescribed values of ρ and B. Then the ratio of it to its formal series expansion serves as a second-order multiplicative correction factor. Hence a uniformly valid second approximation is given by

$$\frac{\overline{q}_2}{U} = \left[\frac{\text{exact solution}}{1+\left[-\frac{\rho}{4z}+R_1(z)\right]+\left\{\frac{M^2}{4}\frac{\rho^2}{z^2}\ln\frac{\beta^2\rho}{2z}+\left(\frac{3}{32}+\frac{3}{8}M^2-\frac{M^2 n}{4}\right)\frac{\rho^2}{z^2}-\left[\frac{3}{16}B+\frac{R_1(0)}{4}\right]\frac{\rho}{z}+R_2(z)\right\}}\right]_{\substack{\text{semi-}\\\text{infinite}\\\text{body}}} \frac{"q"_2}{U} \tag{60a}$$

Dividing both the series for $"q_2"/U$ and that in the denominator of the bracket by equation (59) makes them regular at $z = 0$, and gives, after expansion,

$$\frac{\overline{q}_2}{U} = \left[\frac{\text{exact solution}}{1+R_1(z)+\left\{R_2(z)+\frac{1}{4}\left[R_1(z)-R_1(0)\right]\frac{\rho}{z}\right\}}\right]_{\substack{\text{semi-}\\\text{infinite}\\\text{body}}} \left[\frac{"q_2"}{U}+\frac{\rho}{4z}\frac{"q_1"}{U}+\right.$$

$$\left.\left(\frac{M^2}{4}\ln\frac{2z}{\beta^2\rho}-\frac{1}{32}-\frac{3}{8}M^2+\frac{M^2 n}{4}\right)\frac{\rho^2}{z^2}+\frac{3}{16}B\frac{\rho}{z}\right] \tag{60b}$$

This rule may be written finally as[6]

[6]As $M \to 0$ this reduces not to the rule of equation (42b) but to an equally valid alternative.

$$\frac{\overline{q}_2}{U} = Q\left(\frac{2z}{\rho}, B; M\right)\left[\frac{"q_2"}{U} + \frac{\rho}{4z}\frac{"q_1"}{U} + \left(\frac{M^2}{4}\ln\frac{2z}{\beta^2\rho} - \frac{1}{32} - \frac{3}{8}M^2 + \frac{M^2 n}{4}\right)\frac{\rho^2}{z^2} + \frac{3}{16}B\frac{\rho}{z}\right] \tag{60c}$$

where $Q(2z/\rho, B; M)$ stands for the first bracket in equation (60b). Note that the terms in the bracket are just those required to cancel all singularities in $"q_2"/U$.

A combined rule for two round ends is again found by applying the rule twice in succession and simplifying insofar as possible. The result is

$$\begin{aligned}\frac{\overline{q}_2}{U} = {} & Q\left[\frac{2(x-a)}{\rho_a}, B_a; M\right] Q\left[\frac{2(b-x)}{\rho_b}, B_b; M\right]\left\{\frac{"q_2"}{U} + \frac{1}{4}\left(\frac{\rho_a}{x-a} + \frac{\rho_b}{b-x}\right)\frac{"q_1"}{U} + \right. \\ & \frac{M^2}{4}\left[\frac{\rho_a^2}{(x-a)^2}\ln\frac{2(x-a)}{\beta^2\rho_a} + \frac{\rho_b^2}{(b-x)^2}\ln\frac{2(b-x)}{\beta^2\rho_b}\right] - \frac{1}{32}\left(\frac{\rho_a}{x-a} - \frac{\rho_b}{b-x}\right)^2 + \\ & \left. M^2\left(\frac{n}{4} - \frac{3}{8}\right)\left[\frac{\rho_a^2}{(x-a)^2} + \frac{\rho_b^2}{(b-x)^2}\right] + \frac{3}{16}\left(\frac{B_a\rho_a}{x-a} + \frac{B_b\rho_b}{b-x}\right)\right\}\end{aligned} \tag{60d}$$

Mixed Rules Based on the Paraboloid

The function $Q(2z/\rho, B; M)$ required in the above rules could in principle be determined to any desired accuracy by the Janzen-Rayleigh approximation. However, the practical details appear almost insurmountable except in the special case of the paraboloid, for which $B = 0$. (The next section shows that even the solution for the paraboloid has not yet been carried far enough to yield reasonable accuracy at high Mach numbers.) It is therefore worthwhile to simplify the rules so as to base them on the solution for the paraboloid.

If B is to be replaced by zero in the argument of Q, it must be omitted elsewhere in equations (60). Hence the rule for a single round edge, corresponding to equation (60c), is

$$\frac{\overline{q}_{2,1}}{U} = Q\left(\frac{2z}{\rho}; M\right)\left[\frac{"q_2"}{U} + \frac{\rho}{4z}\frac{"q_1"}{U} + \left(\frac{M^2}{4}\ln\frac{2z}{\beta^2\rho} - \frac{1}{32} - \frac{3}{8}M^2 + \frac{M^2 n}{4}\right)\frac{\rho^2}{z^2}\right] \tag{61a}$$

Likewise the counterpart of equation (60d) for two round edges is

$$\frac{\bar{q}_{2,1}}{U} = Q\left[\frac{2(x-a)}{\rho_a}\ ;\ M\right] Q\left[\frac{2(b-x)}{\rho_b}\ ;\ M\right]\left\{\frac{"q_2"}{U} + \frac{1}{4}\left(\frac{\rho_a}{x-a} + \frac{\rho_b}{b-x}\right)\frac{"q_1"}{U} + \right.$$

$$\frac{M^2}{4}\left[\frac{\rho_a^2}{(x-a)^2}\,ln\,\frac{2(x-a)}{\beta^2\rho_a} + \frac{\rho_b^2}{(b-x)^2}\,ln\,\frac{2(b-x)}{\beta^2\rho_b}\right] - \frac{1}{32}\left(\frac{\rho_a}{x-a} - \frac{\rho_b}{b-x}\right)^2 +$$

$$\left. M^2\left(\frac{n}{4} - \frac{3}{8}\right)\left[\frac{\rho_a^2}{(x-a)^2} + \frac{\rho_b^2}{(b-x)^2}\right]\right\} \tag{61b}$$

These rules give a result that is, of course, correct to second order except near the ends. It is correct only to first order within a distance of the order of the radius ρ from the ends. Finally, it is completely invalid within a much smaller neighborhood of the ends of the order of $B^2\rho$ (which is proportional to the sixth power of the thickness ratio for a body of unit length). The reason is that the bracket in equation (61a) has not been completely freed of singularities, but retains a term $-3B\rho/16z$ (which is cancelled in the original rule of eq. (60c)). For most practical purposes this distance is so minute as to be altogether negligible, and (as indicated by the subscripts) these mixed rules can be regarded as yielding a solution valid to first order near the ends and to second order elsewhere.

Example: Uniformly Valid Solution for Ellipsoid

These rules can now be applied to the formal solution of equation (54b) to find a uniform approximation for the ellipsoid. Using the combined rule of equation (60d) gives as the uniform second-order solution

$$\frac{\bar{q}_2}{U} = Q\left[\frac{2(1+x)}{\tau^2}\ ,\ \tau^2;\ M\right] Q\left[\frac{2(1-x)}{\tau^2}\ ,\ \tau^2;\ M\right]\left\{1 + \tau^2\left(ln\,\frac{2}{\beta\tau} - \frac{1}{2}\right) + \right.$$

$$\left. \tau^4\left[(1-M^2)\,ln^2\,\frac{2}{\beta\tau} + 2M^2\,ln\,\frac{2}{\beta\tau} + M^2 n - \frac{3}{8} - \frac{3}{2}M^2\right]\right\} \tag{62}$$

The last bracket has of course been freed of singularities, but in addition it is seen to be a constant, independent of x. This simple feature is the counterpart, for second-order subsonic flow, of Munk's rule for incompressible flow past an ellipsoid, according to which the surface velocity is just the projection of the maximum velocity.

Using instead the mixed rule of equation (61b) yields

$$\frac{\bar{q}_{2,1}}{U} = Q\left[\frac{2(1+x)}{\tau^2}\ ;\ M\right] Q\left[\frac{2(1-x)}{\tau^2}\ ;\ M\right]\left\{1+\tau^2\left(\ln\frac{2}{\beta\tau}-\frac{1}{2}\right)+\right.$$

$$\left.\tau^4\left[(1-M^2)\ln^2\frac{2}{\beta\tau}+2M^2\ln\frac{2}{\beta\tau}+M^2 n-\frac{3}{8}-\frac{3}{2}M^2-\frac{3}{8}\frac{1}{1-x^2}\right]\right\} \quad (63)$$

and the remaining nose singularity is evident. Its effect may be illustrated in the case of incompressible flow. At a distance of one radius from the nose (which would be 0.02 of the length for a 20-percent-thick ellipsoid), equation (63) gives

$$\frac{\bar{q}_{2,1}}{U} = \sqrt{\frac{2}{3}}\left[1+\tau^2\left(\ln\frac{2}{\tau}-\frac{13}{16}\right)\right]$$

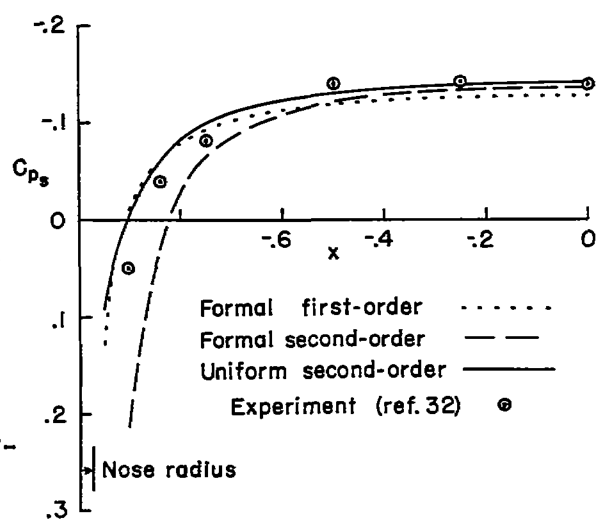
The exact solution has -12/16 instead of -13/16, showing that the result is valid only to first (though nearly to second) order. Again, at a distance of only $B\rho = \tau^4$ from the end (0.0008 of the length), equation (63) gives

$$\frac{\bar{q}_{2,1}}{U} = \frac{13}{16}\sqrt{2}\ \tau\left(1+\ .\ .\ .\right)$$

The exact solution lacks the factor 13/16, but the leading term in the pressure coefficient is nevertheless given correctly as unity, so the result is regarded as being correct to first order. This ceases to be true only at distances of the order of $B^2\rho$ from the end, which is 0.000032 of the length for a 20-percent-thick ellipsoid.

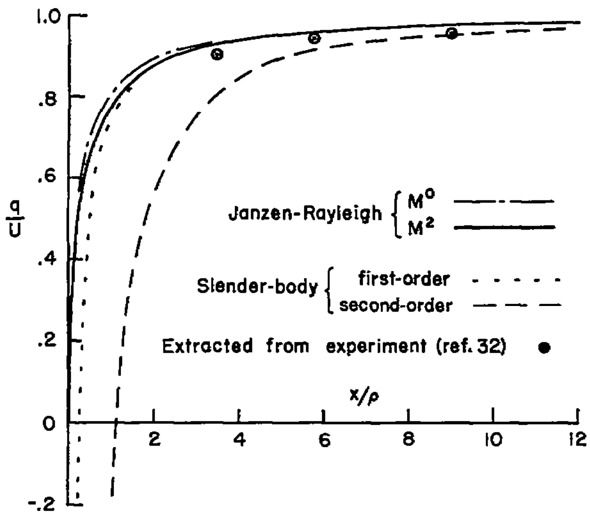
Comparison With Experiment

Matthews (ref. 32) has measured pressures over the front half of an ellipsoid of revolution of thickness ratio 1/6 up to Mach numbers of 0.940 (the measured critical Mach number being 0.916). The ellipsoid was supported from the rear by a sting but, according to first-order slender-body theory, the sting affects the pressure coefficient over the front half by less than 0.003, which is negligible. The pressures measured at $M = 0.900$ are compared with first- and second-order theory in sketch (n).



Sketch (n).- Pressure on ellipsoid of revolution with $\tau = 1/6$ at $M = 0.900$.

Over the middle of the body the experimental values agree closely with the second approximation, which is seen to be a significant improvement over the first. Near the ends, however, the experimental values lie between the predictions of second-order theory with and without the application of the mixed rule. The reason for this is believed to be simply that the values of Q, the surface speed on a paraboloid, used in the rule are inaccurate. They were taken from the Janzen-Rayleigh approximation including only terms in M^2 (table II of ref. 14), which is almost certainly inadequate at M = 0.900. Indeed, the present theoretical results are believed to be sufficiently trustworthy that one can work backwards to extract experimental values of Q for the paraboloid from the measurements on ellipsoids. The result is shown in sketch (o) in comparison with all existing theories.



Sketch (o).- Speed ratio on paraboloid of revolution at M = 0.90.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., May 28, 1958

APPENDIX A

PRINCIPAL SYMBOLS

a	abscissa of nose of body
b	abscissa of tail of body
B	$(M^2 - 1)^{1/2}$
$\mathcal{B}$	bluntness of nose (see eq. (42a))
C	constant in equations (58)
C_p	pressure coefficient
c	half-length of body
D	drag
F	source strength in slender-body theory, RR'
f	second-order increment in F
$G(x)$	term independent of r in slender-body potential
$g(x)$	second-order increment in $G(x)$
I_a^b	subsonic slender-body integral (see eq. (14a))
J_a^x	supersonic slender-body integral (see eq. (14b))
k	arbitrary constant
K	constant term in slender-body potential (see eqs. (47) and (52))
l	length of body
$L_2(x)$	Euler's dilogarithm
M	free-stream Mach number
n	$\frac{\gamma + 1}{2} \frac{M^2}{1 - M^2}$
N	supersonic counterpart of n, $\frac{\gamma + 1}{2} \frac{M^2}{M^2 - 1}$

P, P_1, P_2 p_1, p_2, p_3	general functions in similarity rules (see eqs. (22) and (28))
Q	speed on surface of semi-infinite round-nosed body, referred to free-stream speed
q	local speed of flow
$R(x)$	radius of meridian curve of body of revolution
R_1, R_2	regular functions
r	radius in cylindrical polar coordinates
U	free-stream speed
u	streamwise velocity component
x	streamwise coordinate
z	abscissa measured from round end into body
β	$(1 - M^2)^{1/2}$
γ	adiabatic exponent
Δ_2	second-order increment
δ	initial slope of sharp-nosed body
ϵ	small parameter
λ	(See eq. (42b).)
ρ	nose radius of round-nosed body
ρ_∞	free-stream density
τ	thickness ratio
Φ	full velocity potential
φ	first-order perturbation velocity potential
ϕ	second-order perturbation velocity potential

Subscripts and Superscripts

$(\ )_a$	associated with nose
$(\ )_b$	associated with tail
$(\ )_s$	value on surface of body
$(\ )_1$	first-order value
$(\ )_2$	second-order value
$(\ )_{2,1}$	mixed second- and first-order value
$(\ )_{max}$	maximum value
$(\ )_*$	regular part (see eq. (52b))
$(\tilde{\ })$	singular part (see eq. (51))
$(\ )'$	derivative
$(\bar{\ })$	uniformly valid value
"( )"	formal value

APPENDIX B

SHORT TABLE OF SLENDER-BODY INTEGRALS

The integrals appearing in the slender-body solution were denoted in equations (14) by

$$I_a^b\left\{F(x)\right\} \equiv \int_a^b \frac{F(x) - F(\xi)}{|x - \xi|}\, d\xi \qquad \text{(subsonic)} \qquad \text{(B1a)}$$

$$J_a^x\left\{F(x)\right\} \equiv \int_a^x \frac{F(x) - F(\xi)}{x - \xi}\, d\xi \qquad \text{(supersonic)} \qquad \text{(B1b)}$$

In the supersonic case the notation is designed to emphasize the different roles played by x in the integrand and in the upper limit of integration. The subsonic integral can be expressed in terms of the supersonic one by

$$I_a^b\left\{F(x)\right\} = \operatorname{sgn}(x - a) J_a^x\left\{F(x)\right\} + \operatorname{sgn}(b - x) J_b^x\left\{F(x)\right\} \qquad \text{(B2)}$$

of which equation (16) is a special case.

For purposes of shifting the origin of abscissas, it is convenient to relate the general supersonic integral to that for some standard value of the lower limit a, say zero. The desired expression is easily seen to be[1]

$$J_a^x\left\{F(x)\right\} = J_0^{x-a}\left\{F(x + a)\right\} \qquad \text{(B3)}$$

Combining these last two results gives a useful expression for the general subsonic integral in terms of the standard supersonic one; for $a \leq x \leq b$,

$$I_a^b\left\{F(x)\right\} = J_0^{x-a}\left\{F(x + a)\right\} + J_0^{x-b}\left\{F(x + b)\right\} \qquad \text{(B4)}$$

A short table of the subsonic and supersonic slender-body integrals is given below. The limits of integration have been taken as $a = -1$, $b = 1$ for the subsonic case and $a = 0$ for the supersonic. Results for

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[1] The meaning of $J_0^{x-a}\left\{F(x + a)\right\}$ in conjunction with the following table is that one looks up $F(x + a)$ in the column labeled $F(x)$ and in the corresponding column labeled J_0 replaces x by $(x - a)$.

other ranges of integration can be extracted using the two relations above (eqs. (B3) and (B4)). Values of the subsonic integral are given only for the span of the body, that is, for $-1 \leq x \leq 1$. The table was checked where possible using equation (B4).

$F(x)$	$I_{-1}^{1}\left\{F(x)\right\} \equiv \int_{-1}^{1}\frac{F(x)-F(\xi)}{\lvert x-\xi\rvert}d\xi,\ \lvert x\rvert \leq 1$	$J_{0}^{x}\left\{F(x)\right\} \equiv \int_{0}^{x}\frac{F(x)-F(\xi)}{x-\xi}d\xi$
1	0	0
x	$2x$	x
x^2	$3x^2-1$	$\frac{3}{2}x^2$
x^3	$\frac{11}{3}x^3-x$	$\frac{11}{6}x^3$
x^4	$\frac{25}{6}x^4-x^2-\frac{1}{2}$	$\frac{25}{12}x^4$
$x^n, n=1,2,\ldots$	$2\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)x^n-\left[x^{n-2}+\frac{1}{2}x^{n-4}+\ldots+\frac{1+(-)^{n-1}}{n-1}x+\frac{1+(-)^n}{n}\right]$	$\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)x^n$
$\sqrt{x}$	- - -	$2(1-\ln 2)\sqrt{x}$
$\sqrt{1-x^2}$	$2\left[\sqrt{1-x^2}\left(1-\ln 2\sqrt{1-x^2}\right)-x\sin^{-1}x\right]$	- - -
$\ln x$	- - -	$\frac{\pi^2}{6}$
$x\ln x$	- - -	$x\left(\ln x+\frac{\pi^2}{6}-1\right)$
$x^2\ln x$	- - -	$x^2\left(\frac{3}{2}\ln x+\frac{\pi^2}{6}-\frac{5}{4}\right)$
$\ln(1-x)$	$\frac{\pi^2}{6}-L_2\left(\frac{1+x}{2}\right)-\frac{1}{2}\ln^2\frac{1-x}{2}$	$-L_2(x)-\frac{1}{2}\ln^2(1-x)$
$x\ln(1-x)$	- - -	$-xL_2(x)-\frac{1}{2}x\ln^2(1-x)-(1-x)\ln(1-x)-x$
$x^2\ln(1-x)$	- - -	$-x^2L_2(x)-\frac{1}{2}x^2\ln^2(1-x)-\frac{1}{2}(1-x)(1-2x)\ln(1-x)-\frac{1}{2}x-\frac{5}{4}x^2$
$x^3\ln(1-x)$	- - -	$-x^3L_2(x)-\frac{1}{2}x^3\ln^2(1-x)-\frac{1}{6}(1-x)(2+5x+11x^2)\ln(1-x)-\left(\frac{1}{3}x+\frac{2}{3}x^2+\frac{49}{36}x^3\right)$
$x^4\ln(1-x)$	- - -	$-x^4L_2(x)-\frac{1}{2}x^4\ln^2(1-x)-(1-x)\left(\frac{1}{4}+\frac{7}{12}x+\frac{13}{12}x^2+\frac{25}{12}x^3\right)\ln(1-x)-\left(\frac{1}{4}x+\frac{11}{24}x^2+\frac{3}{4}x^3+\frac{205}{144}x^4\right)$
$\ln(1-x^2)$	$\frac{\pi^2}{6}-\frac{1}{2}\ln^2\frac{1+x}{1-x}$	- - -
$x^2\ln(1-x^2)$	$x^2\left(\frac{\pi^2}{6}-\frac{1}{2}\ln^2\frac{1+x}{1-x}\right)+2x\ln\frac{1+x}{1-x}+(3x^2-1)\ln(1-x^2)+(1-5x^2)$	- - -
$x^4\ln(1-x^2)$	$x^4\left(\frac{\pi^2}{6}-\frac{1}{2}\ln^2\frac{1+x}{1-x}\right)+2x\left(\frac{1}{3}+x^2\right)\ln\frac{1+x}{1-x}+\left(\frac{25}{6}x^4-x^2-\frac{1}{2}\right)\ln(1-x^2)+\left(\frac{3}{4}-\frac{5}{6}x^2-\frac{205}{36}x^4\right)$	- - -

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